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DYNAMIC PLASTIC BEHAVIOR OF SPHERICAL SHELL

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DYNAMIC PLASTIC BEHAVIOR OF SPHERICAL SHELL

by

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SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
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by

Nguyên-Tiên-Ich

Submitted to the Department of Naval Architecture and Marine Engineering on May 7, 1971, in partial fulfillment of the requirement for the degree of Doctor of Philosophy.

ABSTRACT

A dynamic pressure decaying exponentially with time, whose analytic representation in terms of non-dimensional quantities is:

$$p = p_o e^{-\tau} \qquad (\tau \geq 0) \ ,$$

and whose peak value p_o is larger than the static collapse pressure p_s, is applied uniformly throughout a spherical shell made of rigid-perfectly plastic material. The problem is to determine the final deflection of the shell for a given yield surface, the shell has no holes and is simply supported at the outer edge.

Three approximate yield surfaces have been used: the two-moment limited interaction (HS), the uncoupled square (SS) and the uncoupled diamond (DS) yield surfaces.

For each yield surface selected, the static collapse pressure is first determined both for simply supported and clamped edges. In the dynamic problem, the shell is simply supported.

The first part deals with shallow shells. With the (HS) and (SS) yield surfaces, solutions for three ranges of pressure have been obtained. With the (DS) yield surface, solution for one range of pressure has been obtained.

The second part deals with deeper spherical caps. For all three yield surfaces, only solutions for one range of pressure have been obtained.

For shallow shells, graphs giving maximum central deflection and energy absorbed as functions of the pressure difference p_o-p_s and maximum central deflection as functions of energy absorbed are also presented.

Thesis Supervisor: Professor N. Jones

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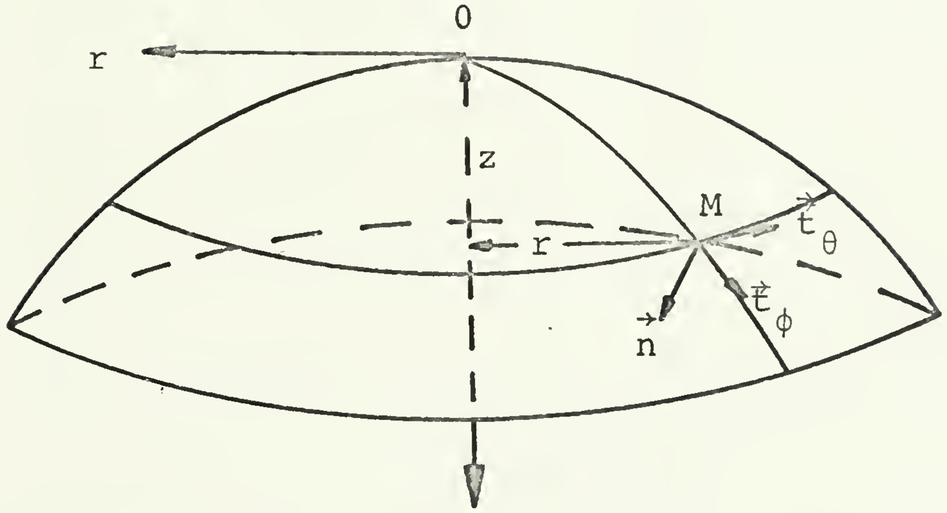
NOTATION



Global and Local Coordinates of the Reference Surface

Fig. 1

0z: Axis of symmetry.

M: Generic point on the reference surface, which is the mid-surface for a homogeneous shell of constant thickness.

r: Distance from M to the axis of symmetry 0z.

z: Distance from M to the tangent plane at the apex 0.

$\vec{n}$: Unit normal to the reference surface, positive inward.

$\vec{t}_\phi$: Unit tangent to the meridian at M, pointing away from the apex 0.

$\vec{t}_\theta$: Unit tangent to the parallel at M.

$$\vec{t}_\theta = \vec{t}_\phi \times \vec{n}$$

U_ϕ: Displacement component in the $\vec{t}_\phi$ direction.

U_n: Displacement component in the $\vec{n}$ direction.

A: A reference length.

T_o: A reference time.

$v = \frac{U_\phi}{A}$: Non-dimensional meridional component of the displacement.

$w = \frac{U_n}{A}$: Non-dimensional normal component of the displacement.

t: Time variable.

$\tau = \frac{t}{T_o}$: Non-dimensional time variable.

$x = \frac{r}{A}$: Non-dimensional space variable.

$y = \frac{z}{A}$: Non-dimensional space variable.

$\dot{v} = \frac{\partial v}{\partial \tau}$: Non-dimensional meridional component of the velocity.

$\dot{w} = \frac{\partial w}{\partial \tau}$: Non-dimensional normal component of the velocity.

$\dot{e}_\theta$: Non-dimensional strain rate in the $\vec{t}_\theta$ direction.

$\dot{e}_\phi$: Non-dimensional strain rate in the $\vec{t}_\phi$ direction.

$\dot{k}_\theta$: Non-dimensional curvature rate in the $\vec{t}_\theta$ direction.

$\dot{k}_\phi$: Non-dimensional curvature rate in the $\vec{t}_\phi$ direction.

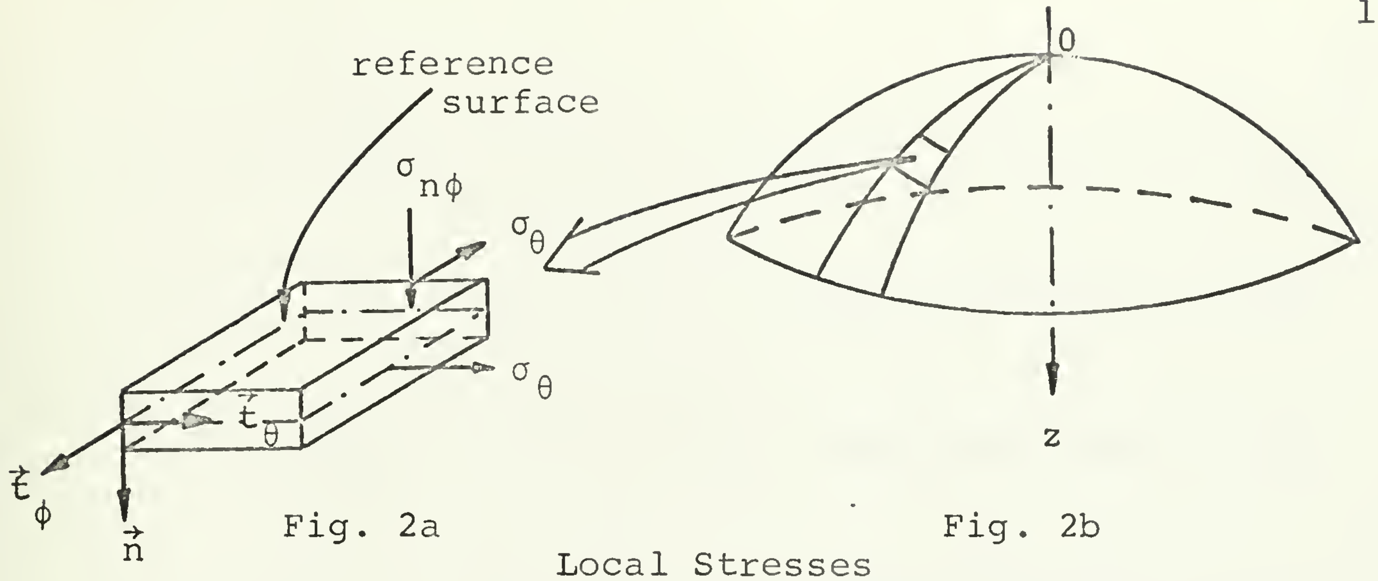


Fig. 2a Fig. 2b

Local Stresses

σ_θ: Membrane stress in the $\vec{t}_\theta$ direction.

σ_ϕ: Membrane stress in the $\vec{t}_\phi$ direction

$\sigma_{n\phi}$: Transverse shear stress.

2H: Shell thickness.

ξ: Distance from a point of the shell to the reference surface, positively inward.

For thin shell, for which $\frac{H}{R_p} << 1$, where R_p is the smaller principal radius of curvature of the reference surface, the stress and moment resultants per unit length are:

$$N_\theta = \int_{-H}^{+H} \sigma_\theta d\xi \quad , \quad N_\phi = \int_{-H}^{+H} \sigma_\phi d\xi \quad , \quad S = \int_{-H}^{+H} \sigma_{n\phi} d\xi$$

$$M_\theta = \int_{-H}^{+H} \sigma_\theta \xi d\xi \quad , \quad M_\phi = \int_{-H}^{+H} \sigma_\phi \xi d\xi$$

σ_o: Yield stress.

N_o: $\sigma_o \times 2H$

M_o: $\sigma_o H^2$

$n_\theta = \frac{N_\theta}{N_o}$: Non-dimensional stress resultant in the $\vec{t}_\theta$ direction.

$n_\phi = \frac{N_\phi}{N_o}$: Non-dimensional stress resultant in the $\vec{t}_\phi$ direction.

$s = \frac{S}{N_o}$: Non-dimensional transverse shear stress resultant.

$m_\theta = \frac{M_\theta}{M_o}$: Non-dimensional moment resultant of the σ_θ-stresses.

$m_\phi = \frac{M_\phi}{M_o}$: Non-dimensional moment resultant of the σ_ϕ-stresses.

$\dot{D}_{int}$: Rate of internal energy dissipation.

$\dot{D}_{ext}$: Rate of external energy input.

$\dot{d}_{int} = \frac{\dot{D}_{int}}{\frac{2\pi N_o A^2}{T_o}}$: Non-dimensional rate of internal energy dissipation.

$\dot{d}_{ext} = \frac{\dot{D}_{ext}}{\frac{2\pi N_o A^2}{T_o}}$: Non-dimensional rate of external energy input.

ρ: Surface density of the shell material.

P: Applied external pressure, in the normal direction.

$p = \frac{AP}{N_o}$: Non-dimensional pressure applied externally in the normal direction.

$\bar{s}$: Non-dimensional applied traction at the edge, in the $\vec{n}$-direction.

$\bar{n}_\phi$: Non-dimensional applied traction at the edge, in the $\vec{t}_\phi$-direction.

$\bar{m}_\phi$: Non-dimensional applied moment at the edge.

$\gamma = \frac{\rho A^2}{N_o T_o^2}$: Non-dimensional surface density of the shell material.

1. INTRODUCTION

The first paper on the dynamic plastic behavior of a beam was published in the early 50's by E. H. Lee and P. S. Symonds [1]. In this problem and in those by subsequent authors [2, etc.], the material is assumed to be rigid perfectly plastic and the vibrations of the structures are neglected so that the problem of unloading has not to be considered. These assumptions, which simplify problems considerably, do not introduce important errors in many cases, since on the one hand dynamic energies are sometimes much larger than the elastic energy, and on the other hand the active duration of a pressure pulse is often much smaller than the corresponding natural time period of the structure.

Theoretical investigations into the dynamic behavior of axisymmetric plates and shells were initiated with a paper published on circular plates by H. G. Hopkins and W. Prager [3]. Various boundary, geometrical and loading conditions have been considered: H. G. Hopkins and W. Prager solved the problem of simply supported circular plates under uniform dynamic load of rectangular profile [3]; A. J. Wang gave a solution of simply supported circular plates under impulsive loading [4]. Recently, M. F. Conroy published a paper on simply supported circular plates due to

dynamic circular loading [16]. The problem of clamped circular plates was treated by A. J. Wang and H. G. Hopkins for impulsive load [5] and by A. L. Florence for a uniform rectangular pulse load type [6]. A. L. Florence also solved the problem of clamped plates subjected to a rectangular pulse load type uniformly distributed over a central circular area [7]. The problem of an annular plate clamped along its inner edge and free along its outer one and subjected to impulsive load was examined by G. S. Shapiro [8].

In all these problems, the maximum displacement was supposed to be infinitesimal so that the strain rate-velocity relations are linear and only the bending moments are considered to be important. N. Jones examined the problem of finite deflection of simply supported circular plates under impulsive load and developed a theoretical procedure which retains both membrane forces and bending moments [9]. The solutions compare favorably with some experimental results on simply supported circular plates.

Although the behavior of circular plates has been considered in some detail, there are few solutions available for the axisymmetric behavior of shells. P. G. Hodge studied the rigid plastic behavior of an infinitely long cylindrical shell reinforced with equally spaced rigid rings uniformly loaded with a rectangular pressure profile [10].

Another paper by P. G. Hodge with a numerical solution, examined the influence of blast characteristics on the final deformation of circular cylindrical shells [11].

In plasticity, the choice of a yield surface is of crucial importance. In most of the problems dealing with plates, the Tresca yield surface, because of its simplicity, has been adopted. However, for shells the Tresca yield surface becomes quite complicated. Although Tresca yield surface for axisymmetric shells has been developed by E. T. Onat and W. Prager [12], it has been used rather as a reference surface. To make shell problems tractable analytically, some simplified yield surface must be devised. In static plasticity, the general theorems of limit analysis allow one to bracket the exact value of the collapse pressure and to evaluate the degree of accuracy of an approximate yield surface. P. G. Hodge and B. Paul have considered the influence of approximate yield conditions in the dynamic behavior of a simply supported cylindrical shell subjected to uniform pressure pulses having various profiles [13]. Although they made some useful suggestions, no general theorems corresponding to the limit theorems have been established with respect to the predictions of approximate yield surfaces.

The spherical shell which has two equal curvatures is the next shell geometry that was considered. R. San-

karanarayanan obtained the solution to the problem of spherical cap subjected to uniform dynamic load which decays exponentially [14] or has a rectangular profile [15]. The results obtained are involved and a numerical approach had to be used to examine the question of admissibility.

It has been observed in elasticity that problems are usually simpler when analyzed as shallow shells and that the results obtained may give some indications of the corresponding behavior of deep shells. This study starts with dynamic plasticity of shallow spherical shells. Analytical procedures have been used as far as possible and three kinds of yield surfaces have been used, of which two only give distinct solutions. Then general shells have also been examined. The solutions obtained are complicated, for in the cases of plates, except for [9], and of cylindrical shells, there are two general stresses and one velocity component, for which there is one equilibrium equation; the other two equations can be found from the yield surface and the flow rule. In the case of spherical shells, there are five general stresses and two velocity components, for which there are three equilibrium equations and the other four equations can be found from the yield surface and the flow rules. It is obvious that difficulties increase with the number of unknowns and the order of the equations.

2. SHALLOW SPHERICAL SHELLS

2.1 General Relations

2.1.1 Strain rate-velocity relations

For rotationally symmetrical shallow shells [17] loaded symmetrically, the relations between the generalized strain rates and velocities when it is assumed that $y'^2_{max} \ll 1$, y' being the slope of the meridian of the shell, are:

$$\dot{e}_\theta = \frac{\dot{v} - y'\dot{w}}{x} \qquad (2.1.1.1)$$

$$\dot{k}_\theta = -h\,\frac{\dot{w}' + y''\dot{v}}{x} \qquad (2.1.1.2)$$

$$\dot{e}_\phi = \dot{v}' - y''\dot{w} \qquad (2.1.1.3)$$

$$\dot{k}_\phi = -h(\dot{w}' + y''\dot{v})' \qquad (2.1.1.4)$$

with $( )' = \dfrac{d( )}{dx}$

and $$h = \frac{M_o}{AN_o} \qquad (2.1.1.5)$$

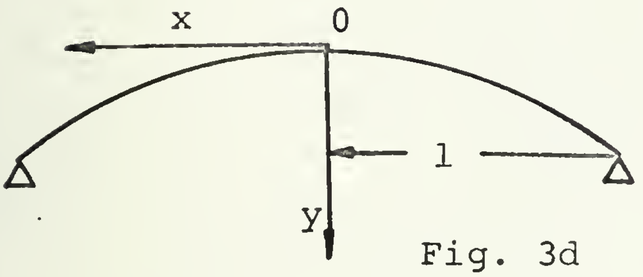


Fig. 3d

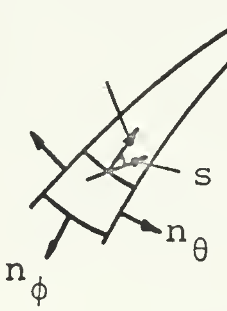


Fig. 3c

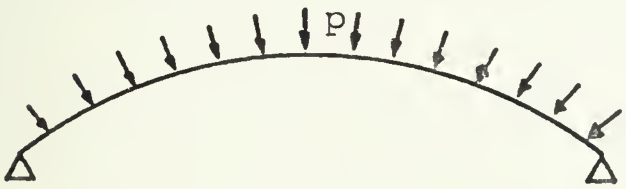


Fig. 3a

Positive Sense of Moment

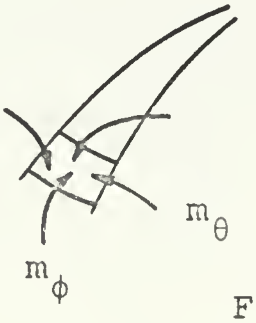


Fig. 3b

In this case, the reference length A is the base radius R of the shell:

$$A = R$$

Then, from (2.1.1.5) page 19:

$$h = \frac{M_o}{RN_o}$$

$$h = \frac{H}{2R} \qquad (2.1.1.6)$$

2.1.2 Equilibrium Equations

Using the results from [17] for shallow symmetrical shells loaded symmetrically, and with some modifications to include the inertia terms, the rates of internal energy dissipation and external energy input are:

$$d_{int} = \int_o^1 (n_\theta \dot{e}_\theta + n_\phi \dot{e}_\phi + m_\theta \dot{k}_\theta + m_\phi \dot{k}_\phi)\,x\,dx$$

$$d_{ext} = \int_o^1 [x(p_n - \gamma\ddot{w})\dot{w} + x(p_\phi - \ddot{v})\dot{v}]\;dx + [x(\bar{s}\dot{w} + \bar{n}_\phi \dot{v}) - h\bar{m}_\phi(\dot{w}' + y''\dot{v})]_{x=1}$$

p_n and p_ϕ are respectively the normal and meridional components of the applied load.

With $\dot{e}_\theta$, $\dot{e}_\phi$, $\dot{k}_\theta$, $\dot{k}_\phi$ from (2.1.1.1)-(2.1.1.4) page 19 and applying the principle of virtual velocity, we obtain the following equations of equilibrium:

$$xs = h[xm_\phi)' - m_\theta] - 2hp_\phi x + \frac{4}{3}\gamma h^2 \ddot{\Omega}_\phi x$$

$$(xn_\phi)' - n_\theta - y''(xs) + p_\phi x = x\gamma\ddot{v}$$

$$y''(xn_\phi) + y'n_\theta + (xs)' + p_n x = x\gamma\ddot{w}$$

where $\dot{\Omega}_\phi = \dot{w}' + y''\dot{v}$ is the rate of slope change or angular velocity. $\dot{\Omega}_\phi$ has a sense of rotation contrary to the sense of positive m_ϕ and the term $\frac{4}{3}\gamma h^2 \ddot{\Omega}_\phi = -i(-\ddot{\Omega}_\phi)$ where i is the non-dimensional moment of inertia of unit area of the shell with respect to the reference surface, represents the rotary inertia.

If the load is applied only in the normal direction as a pressure load, and which is our case, we have: $p_\phi = 0$, $p = p_n$, and if the rotary inertia is neglected, then:

$$xs = h[(xm_\phi)' - m_\theta] \qquad (2.1.2.1)$$

$$(xn_\phi)' - n_\theta - y''(xs) = x\gamma\ddot{v} \qquad (2.1.2.2)$$

$$y''(xn_\phi) + y'n_\theta + (xs)' + px = x\gamma\ddot{w} \qquad (2.1.2.3)$$

For a shell without a hole, simply supported at the edge x = 1, and made of isotropic material, we have the following boundary conditions:

At x = 0:

From isotropy consideration:

$$n_\theta = n_\phi \quad , \quad m_\theta = m_\phi$$

From symmetry consideration:

$$s = 0$$

$$\dot{v} = 0 \quad , \quad \dot{w}' = 0$$ or there will be a hinge

At x=1:

For simply supported edge, not free to move:

$$m_\phi = 0$$

$$\dot{w} = 0 \quad , \quad \dot{v} = 0 \qquad \text{or there will be a hinge}$$

The initial conditions are:

$$\tau = 0 \quad , \quad \dot{v} = v = 0 \quad , \quad \dot{w} = w = 0$$

2.1.3 Yield Surfaces

E. T. Onat and W. Prager have developed a yield surface for a rigid perfectly plastic material that obeys Tresca's yield condition [12]. The complexity of the equation defining the yield condition is such that considerable difficulty may be expected in the solution of all but the most trivial of problems. Therefore simpler approximate yield surfaces have to be devised. Three approximate yield surfaces will be used in this study. These are: the two-moment limited interaction proposed by Hodge [17], the uncoupled square yield surface which is a simplified version of the former and the uncoupled diamond yield surface proposed by N. Jones.

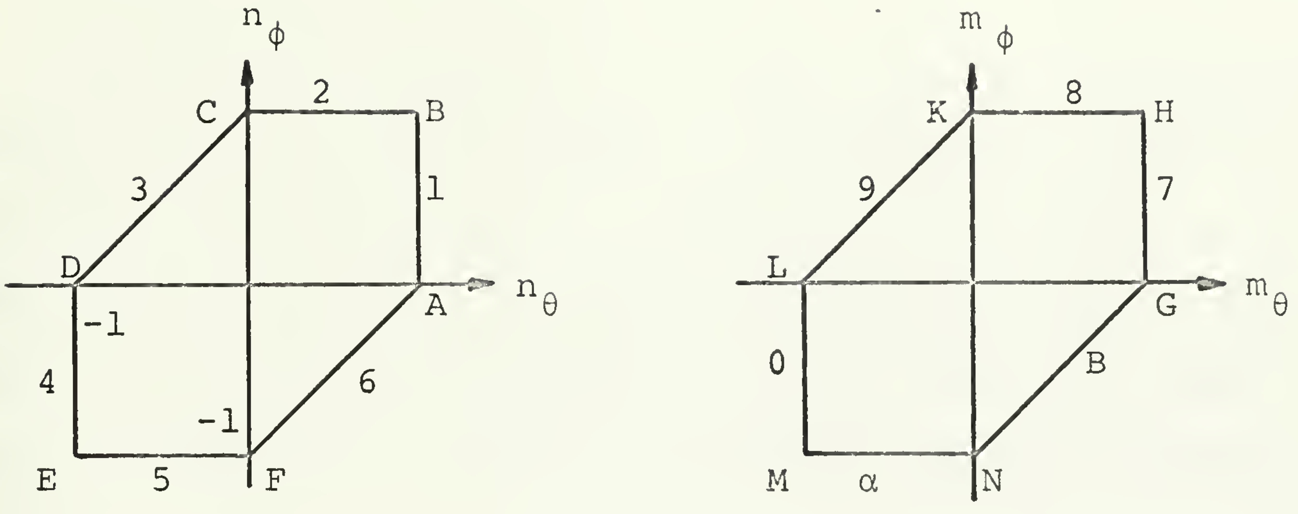


Fig. 4a Fig. 4b

Two-Moment Limited-Interaction Yield Surface

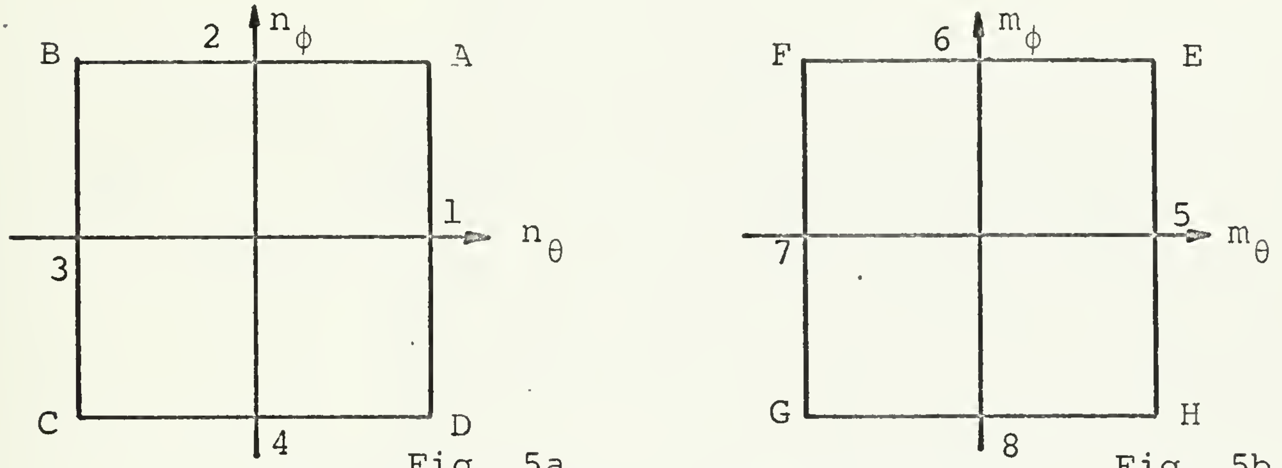


Fig. 5a Fig. 5b

Uncoupled Square Yield Surface

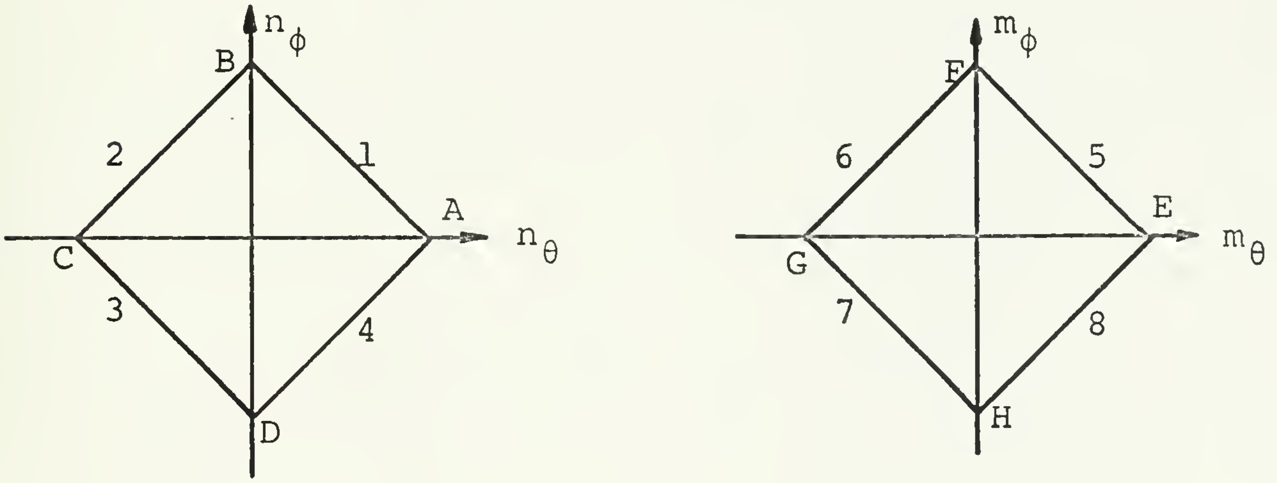


Fig. 6a Fig. 6b

Uncoupled Diamond Yield Surface

The Hodge and Jones yield surfaces allow interaction between force and force or moment and moment but consider interaction between force and moment to be of limited importance. These yield surfaces and the planes that define them and the normality requirements on the associated generalized strain rate vectors are presented in the following tables.

Table 2.1.3.1

Two-Moment Limited-Interaction Surface [17]

Face	Equation	Strain Rate Vector $(\dot{e}_\theta, \dot{e}_\phi, \dot{k}_\theta, \dot{k}_\phi)$
1	$n_\theta = 1$	$\mu(1,0,0,0)$
2	$n_\phi = 1$	$\mu(0,1,0,0)$
3	$-n_\theta+n_\phi = 1$	$\mu(-1,1,0,0)$
4	$-n_\theta = 1$	$\mu(-1,0,0,0)$
5	$-n_\phi = 1$	$\mu(0,-1,0,0$
6	$n_\theta-n_\phi = 1$	$\mu(1,-1,0,0)$
7	$m_\theta = 1$	$\mu(0,0,1,0)$
8	$m_\phi = 1$	$\mu(0,0,0,1)$
9	$-m_\theta+m_\phi = 1$	$\mu(0,0,-1,1)$
0	$-m_\theta = 1$	$\mu(0,0,-1,0)$
α	$-m_\phi = 1$	$\mu(0,0,0,-1)$
β	$m_\theta-m_\phi = 1$	$\mu(0,0,1,-1)$

Table 2.1.3.2

Uncoupled Square Yield Surface

Face	Equation	Strain Rate Vector $(\dot{e}_\theta, \dot{e}_\phi, \dot{k}_\theta, \dot{k}_\phi)$
1	$n_\theta = 1$	$\mu(1,0,0,0)$
2	$n_\phi = 1$	$\mu(0,1,0,0)$
3	$-n_\theta = 1$	$\mu(-1,0,0,0)$
4	$-n_\phi = 1$	$\mu(0,-1,0,0)$
5	$m_\theta = 1$	$\mu(0,0,1,0)$
6	$m_\phi = 1$	$\mu(0,0,0,1)$
7	$-m_\theta = 1$	$\mu(0,0,-1,0)$
8	$-m_\phi = 1$	$\mu(0,0,0,-1)$

Table 2.1.3.3

Uncoupled Diamond Yield Surface

Face	Equation	Strain Rate Vector $(\dot{e}_\theta, \dot{e}_\phi, \dot{k}_\theta, \dot{k}_\phi)$
1	$n_\theta+n_\phi = 1$	$\mu(1,1,0,0)$
2	$-n_\theta+n_\phi = 1$	$\mu(-1,1,0,0)$
3	$-n_\theta-n_\phi = 1$	$\mu(-1,-1,0,0)$
4	$n_\theta-n_\phi = 1$	$\mu(1,-1,0,0)$
5	$m_\theta+m_\phi = 1$	$\mu(0,0,1,1)$
6	$-m_\theta+m_\phi = 1$	$\mu(0,0,-1,1)$
7	$-m_\theta-m_\phi = 1$	$\mu(0,0,-1,-1)$
8	$m\theta-m_\phi = 1$	$\mu(0,0,1,-1)$

Let:

p_{sT} = static collapse pressure of a shallow spherical shell structure according to the Tresca yield surface.

p_{sH} = static collapse pressure of a shallow spherical shell structure according to the two-moment limited-interaction yield surface.

p_{sS} = static collapse pressure according to the uncoupled square yield surface.

p_{sD} = static collapse pressure according to the uncoupled diamond yield surface.

Then according to the theorems of limit analysis (e.g., P. G. Hodge [18]), we can construct the following bounds:

$$0.618\, p_{sH} \leq p_{sT} \leq p_{sH} \qquad (2.1.3.1)$$

$$0.309\, p_{sS} \leq p_{sT} \leq p_{sS} \qquad (2.1.3.2)$$

$$0.618\, p_{sD} \leq p_{sT} \leq 2p_{sD} \qquad (2.1.3.3)$$

These bounds correspond to complete inscribing or circumscribing of the yield surfaces. The interval between the lower and upper bounds can be reduced if only part of the approximate yield surface has to inscribe or circumscribe the exact Tresca yield surface. It will be seen, for instance, that for the uncoupled diamond yield surface, the direct stress resultants of a shallow sphere under the static collapse load have a value of -1/2 approximately. Then, with $n_\theta = -1/2$, $n_\phi = -1/2$, the uncoupled diamond

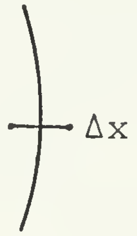
yield surface can be made to circumscribe the corresponding part of the Tresca yield surface by a multiplying factor of 1.5 instead of 2 as stated in (2.1.3.3) page 26.

2.1.4 Discontinuities

From dynamic consideration, it can be shown that s, n_ϕ, m_ϕ must be continuous whereas n_θ, m_θ may be discontinuous [17].

For velocities, we observe that the basic assumption that straight normals to the reference surface of a shell remain straight and normal to the reference surface during deformation implies that shear strains are negligible, and therefore $\dot{w}$ cannot be discontinuous. However w' and $\dot{v}$ may be discontinuous [17]. When such discontinuities occur, we can see, from the expressions (2.1.1.1)-(2.1.1.4) page 19, that the strain rate vector has the following direction:

$$\begin{Bmatrix} \dot{e}_\theta \\ \dot{e}_\phi \\ \dot{k}_\theta \\ \dot{k}_\phi \end{Bmatrix} = \frac{1}{\Delta x} \begin{Bmatrix} 0 \\ \Delta \dot{v} \\ 0 \\ -h(\Delta \dot{w}' + y'' \Delta \dot{v}) \end{Bmatrix} \quad (2.1.4.1)$$



line of discontinuity

Δx is the increase in x to get from one side of the discontinuity line to the other and ΔF is the jump in F corresponding to Δx.

2.2 Dynamic Plastic Response of Simply Supported Shallow Spherical Shells made of Rigid Plastic Material that Obeys the Two-Moment Limited-Interaction Yield Surface

The applied dynamic pressure is assumed to have the form of an exponentially decaying pulse:

$$P = P_o e^{-t/T_o}$$

or, with non-dimensional quantities:

$$p = p_o e^{-\tau}$$

We will solve the system of equations (2.1.2.1)-(2.1.2.3) page 21 using successively the three yield surfaces defined in (2.1.3), page 22 In each case, we begin by determining the static collapse pressure, both for simply supported and clamped supports. A solution is then developed for the dynamic response of a shallow shell with simple supports when the pressure is applied dynamically with a peak value larger than the static collapse pressure.

2.2.1 Static Collapse Pressures

When the pressure is increased slowly from zero so that the inertia forces are negligible, the equations of equilibrium (2.1.2.2), (2.1.2.3) page 21 become:

$$\begin{cases} xs = h[(xm_\phi)' - m_\theta] & (2.2.1.1) \\ (xn_\phi)' - n_\theta - y''xs = 0 & (2.2.1.2) \\ y''(xn_\phi) + y'n_\theta + (xs)' + px = 0 & (2.2.1.3) \end{cases}$$

The boundary conditions for a simply supported edge are:

$$x = 0: \quad s = 0,\ n_\theta = n_\phi,\ m_\theta = m_\phi$$

$$x = 1: \quad m_\phi = 0 \qquad (2.2.1.4)$$

Because of the boundary condition at x=0, a suitable yield regime will be made of one of the faces 1, 2, 4, 5 and one of the faces 7, 8, 0, α. Because of the boundary condition at x=1, only faces 7 or 0 can be retained. Moreover, since the pressure is applied externally, we expect n_ϕ to be negative and m_ϕ to be positive, according to the sign convention in Fig. 3b page 19 Thus the number of choices reduces to two: regime 4-7 or regime 5-7. It turns out that 5-7 is the correct yield regime. (Fig. 4a,b)

With regime 5-7, we have:

$$n_\phi = -1, \qquad \dot{e}_\theta = 0, \ (2.2.1.5) \qquad \dot{e}_\phi \leq 0$$

$$m_\theta = 1, \qquad \dot{k}_\phi = 0, \ (2.2.1.6) \qquad \dot{k}_\theta \geq 0$$

With $n_\phi = -1$, (2.2.1.2) and (2.2.1.3) page 21 become:

$$\begin{cases} -1 - n_\theta - y''xs = 0 & (2.2.1.7) \\ y''x + y'n_\theta + (xs)' + px = 0 & (2.2.1.8) \end{cases}$$

Solving this system and using the shallow shell approximation: $y'^2 << 1$, we get:

$$xs = -\frac{1}{2}px^2 + xy' \quad (2.2.1.10)$$

$$n_\theta = -1 + y''(\frac{1}{2}px^2 - xy') \quad (2.2.1.11)$$

By integrating (2.1.2.1) page 21 between 0 and x, we obtain:

$$hxm_\phi = hx - \frac{1}{6}px^3 + \int_0^x \xi y' d\xi \quad (2.2.1.12)$$

A. Simply Supported Shells

At x=1, $m_\phi = 0$ for simply supported edge. Thus:

$$p = p_s = 6h + 6 \int_0^1 \xi y' d\xi \quad (2.2.1.13)$$

For a flat plate, $y' = 0$, and we find again the collapse pressure of circular flat plate.

With this expression of p, we have thus:

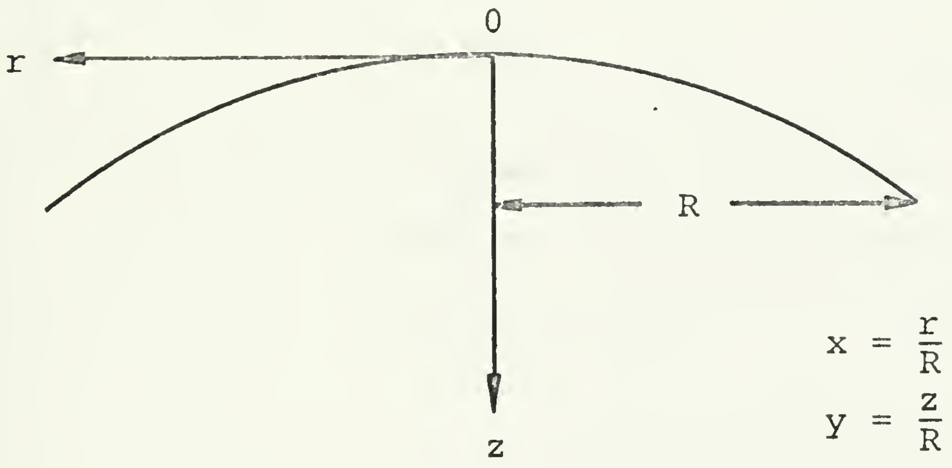


Fig. 7

$$xs = -(3h + 3 \int_0^1 \xi y' d\xi)\, x^2 + xy' \qquad (0 \leq x \leq 1) \quad (2.2.1.14)$$

$$n_\theta = -1 + y''[(3h + 3 \int_0^1 \xi y' d\xi) x^2 - xy'] \qquad (0 \leq x \leq 1) \quad (2.2.1.15)$$

$$m_\phi = 1 - (1 + \frac{1}{h} \int_0^1 \xi y' d\xi) x^2 + \frac{1}{hx} \int_0^x \xi y' d\xi \qquad (0 \leq x \leq 1) \quad (2.2.1.16)$$

The flow rules are given by (2.2.1.5) and (2.2.1.6) page 29. With $\dot{e}_\theta$ and $\dot{k}_\phi$ from (2.1.1.1) and (2.1.1.4) page 19, we have:

$$\dot{v} = y'\dot{w} \qquad (2.2.1.17)$$

$$\dot{w}' + y''\dot{v} = A = \text{constant} \qquad (2.2.1.18)$$

Solving, using the shallow shell approximation and with the boundary condition at x = 1, we have:

$$\dot{w} = \dot{w}_o(1 - x) \qquad (2.2.1.20)$$

and

$$\dot{v} = y'\dot{w}_o(1 - x) \qquad (2.2.1.21)$$

From these velocity expressions, we have:

$\dot{e} = -y'\dot{w}_o \leq 0$ since y' and $\dot{w}_o$ are positive

$$\dot{k} = -\frac{h}{x}\,[-\dot{w}_o + y'y''\dot{w}_o(1 - x)]$$

For a shallow shell, y' is small and y" which is approximately the curvature of the shell is also small, thus $y'y''(1 - x)$ can be considered as small compared to 1 and therefore, we have approximately:

$$\dot{k} = \frac{h}{x}\,\dot{w}_o \geq 0 \quad \text{since } \dot{w}_o > 0$$

The solution is consequently kinematically admissible. We assume the reference surface is smooth so that y' exists everywhere, particularly at the apex x = 0. Because of symmetry, we have then:

$$y'_o = (y')_{x=0} = 0$$

For most practical geometries of cap, the equation of the middle surface can be written in the form [17]:

$$y = \frac{cR^{n-1}}{n} x^n$$

For y' to exist at $x \doteq 0$, we assume $n > 1$.

To determine c, we use the fact that at $x = 1$, we must have $y = Z/R$, where Z is the total depth of the shell. Then:

$$\frac{Z}{R} = c \frac{R^{n-1}}{n} \qquad \text{or}$$

$$c = n \frac{Z}{R^n} \quad ,$$

and the equation of the shallow shell becomes:

$$y = \frac{Z}{R} x^n \qquad n > 1 \qquad (2.2.1.22)$$

$$\text{with } (y'^2)_{\max} = n^2 \frac{Z^2}{R^2} << 1$$

From (2.2.1.22), we have:

$$y' = n\frac{Z}{R} x^{n-1}$$

$$\text{and} \qquad \int_0^x \xi y' d\xi = n \frac{Z}{R} \int_0^x \xi^n d\xi = \frac{n}{n+1} \frac{Z}{R} x^{n+1}$$

With these expressions, (2.2.1.14), (2.2.1.15), and (2.2.1.16) become:

$$xs = -(3h + \frac{3n}{n+1}\frac{Z}{R})\, x^2 + n\frac{Z}{R}\, x^n \qquad (0 \leq x \leq 1) \qquad (2.2.1.23)$$

$$n_\theta = -1 + n(n-1)\frac{Z}{R}\left[(3h + \frac{3n}{n+1}\frac{Z}{R})x^n - n\frac{Z}{R}x^{2(n-1)}\right] \qquad (2.2.1.24)$$

$$(0 \leq x \leq 1)$$

$$m_\phi = 1 - x^2 + \frac{1}{h}\frac{n}{n+1}\frac{Z}{R}(x^n - x^2) \qquad (0 \leq x \leq 1) \qquad (2.2.1.25)$$

From these results, we have:

$$m'_\phi = -2x + \frac{1}{h}\frac{n}{n+1}\frac{Z}{R}(nx^{n-1} - 2x)$$

$$m'_\phi(0) = 0 \qquad \text{and}$$

$$m''_\phi = -2 - \frac{2}{h}\frac{n}{n+1}\frac{Z}{R} + \frac{1}{h}\frac{n}{n+1}\cdot\frac{Z}{R}\, n(n-1)x^{n-2}$$

$$m''_\phi(0) < 0 \qquad \text{for } n \geq 2$$

Thus there is a maximum at x=0:

$$m_\phi(0) = 1 = (m_\phi)_{max}$$

$$m'_\phi = 0 \text{ for } x=0 \qquad \text{and for:}$$

$$x_m^{n-2} = \frac{2}{\frac{1}{h}\frac{n^2}{n+1}\frac{Z}{R}} + \frac{2}{n}$$

If $n \geq 2$ and if

$$\frac{1}{h}\frac{n^2}{n+1}\frac{Z}{R} < 2 + \frac{1}{h}\frac{2n}{n+1}\frac{Z}{R} \qquad \text{or}$$

$$\frac{1}{h}\frac{n(n-2)}{n+1}\frac{Z}{R} < 2\,, \qquad (2.2.1.26)$$

$$(n \geq 2)$$

then $x_m > 1$ and m'_ϕ is negative for $0 \leq x \leq 1$, m_ϕ decreases monotonically from 1 to 0 as x varies from 0 to 1 and is therefore admissible.

If $1 < n < 2$, then $m''_\phi(0)$ goes to positive infinity, which is not allowed since m_ϕ must be maximum at $x = 0$.*

Thus the solution is not admissible for $n < 2$.

For n_θ, we have, from (2.2.1.24):

$$n'_\theta = n^2(n-1)\,\frac{Z}{R}\,x^{n-1}\left[\left(3h + \frac{3n}{n+1}\,\frac{Z}{R}\right) - 2(n-1)\,\frac{Z}{R}\,x^{n-2}\right]$$

For $n \geq 2$, we have:

$$n'_\theta \geq 0 \text{ for } 0 \leq x \leq \left[\frac{3h + \frac{3n}{n+1}\,\frac{Z}{R}}{2(n-1)\frac{Z}{R}}\right]^{\frac{1}{n-2}}$$

---

*For $m_\phi = 1$ to be maximum at $x = 0$, it is necessary that m''_ϕ be negative in the neighborhood of $x = 0$.

If

$$\frac{3h + \frac{3n}{n+1}\frac{Z}{R}}{2(n-1)\frac{Z}{R}} < 1$$

there will be a maximum at

$$x_n = \left[\frac{3h + \frac{3n}{n+1}\frac{Z}{R}}{2(n-1)\frac{Z}{R}}\right]^{\frac{1}{n-2}}$$

To be admissible, this maximum must be negative or from (2.2.1.24) page 33:

$$n(n-1)\frac{Z}{R}x_n^n\,[(3h + \frac{3n}{n+1}\frac{Z}{R}) - n\frac{Z}{R}x_n^{n-2}] < 1$$

or:

$$\frac{n(n-2)}{2}\frac{Z}{R}\,[3h + \frac{3n}{n+1}\frac{Z}{R}]\,x_n^n < 1 \qquad (n \geq 2)$$

Since $x_n \leq 1$ and $3h + \frac{3n}{n+1}\frac{Z}{R} < 3h + \frac{3Z}{R}$, we see that we see that the inequality above will be satisfied if

$$\frac{n(n-2)}{2}\frac{Z}{R}\,[3h + \frac{3Z}{R}] < 1$$

According to Kraus [25, page 25], as a rule of thumb, a shell is assumed thin when its thickness is everywhere less than one-tenth of the radius of curvature of the reference surface. According to an analysis by Reissner quoted by Kraus [25, page 229], a shell is considered as shallow when its maximum height Z is less than one-eighth of its base diameter 2R or:

$$Z/R \leq 1/4$$

From a practical viewpoint, we would expect a shell, even shallow, to have a maximum height Z at least equal to its thickness 2H. Thus, we may assume that

$$0(Z/R) = 0(h) \ ,$$

and:

$$0[\frac{Z}{R}(3h + \frac{3Z}{R}] = 0(\frac{Z^2}{R^2})$$

and the condition

$$\frac{n(n-2)}{2} \frac{Z}{R}(3h + \frac{3Z}{R}) < 1$$

on page 35 is equivalent to

$$\frac{n(n-2)}{2} \times 0(\frac{Z^2}{R^2}) < 1$$

For practical value of n, we can expect that this inequality is satisfied.

At $x = 1$, we must have also:

$$-1 \leq n_\theta \leq 0 \ , \quad \text{or:}$$

$$0 \leq n(n-1) \frac{Z}{R} [3h - \frac{n(n-2)}{n+1} \frac{Z}{R}] \leq 1$$

The condition:

$$0 \leq n(n-1) \frac{Z}{R} [3h - \frac{n(n-2)}{n+1} \frac{Z}{R}]$$

is equivalent to:

$$\frac{1}{h} \frac{n(n-2)}{n+1} \frac{Z}{R} < 3$$

and it is satisfied if (2.2.1.26) page 33 is. The remaining condition, which is

$$n(n-1)\,\frac{Z}{R}\,[3h - \frac{n(n-2)}{n+1}\,\frac{Z}{R}] \leq 1$$

will be satisfied if we have:

$$3n(n-1)\,\frac{Z}{R}\,h \leq 1$$

Since $\frac{Z}{R}\,h$ is of the order of $\frac{Z^2}{R^2}$ which is considered negligible compared to 1, the condition above is expected to be satisfied for practical values of n.

Thus the solution is kinematically and statically admissible for shallow shells of the form:

$$y = \frac{Z}{R}\,x^n \quad , \quad n \geq 2$$

if condition (2.2.2.1.26) page 33 which is:

$$\frac{n(n-2)}{n+1}\,\frac{Z}{R} \leq 2h \qquad (2.2.1.26)$$

is satisfied.

The static collapse pressure is given by (2.2.1.13) page 30, which can be written as:

$$p = p_s = 6h + \frac{6n}{n+1}\,\frac{Z}{R} \quad (n \geq 2) \quad (2.2.1.27)$$

For a shallow surface of second degree, n = 2, which may be a shallow spherical, parabolical, ellipsoidal or hyperboloidal cap, condition (2.2.1.26) is always satisfied and the static collapse pressure in this case is:

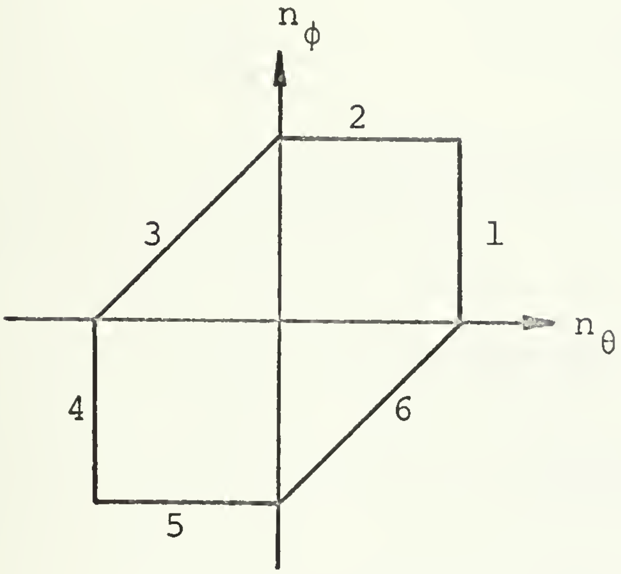
$$p_s = 6h + 4\frac{Z}{R} \qquad (2.2.1.28)$$

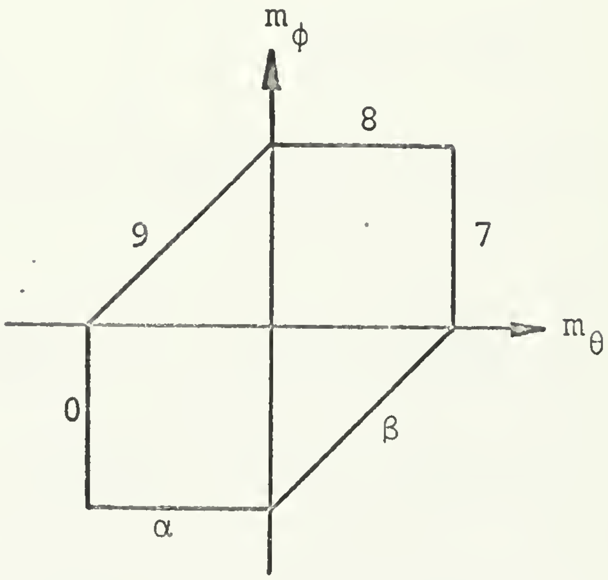
B. Clamped Shells

If the shell is clamped, the boundary condition at $x = 1$ becomes:

$$\dot{w} = 0$$

$\dot{v} = 0$, $\dot{w}' = 0$ or there will be a hinge





We would expect $\dot{w}' \neq 0$ at $x = 1$, and there would be a hinge. From the considerations on discontinuities in 2.1.4, page 27 , we see that this is possible if $|m_\phi| = 1$.

We therefore assume:

$0 \leq x \leq u$: regime 5-7

$u \leq x \leq 1$: regime 5-β

1. Solution for $0 \leq x \leq u$: regime 5-7

For a spherical shallow shell, the value of n in (2.2.1.22) page 32 is 2, and we have:

$$y = \frac{Z}{R} x^2 \qquad (2.2.1.29)$$

then:

$$y' = 2\frac{Z}{R} x \qquad (2.2.1.30)$$

and

$$y'' = \frac{2Z}{R} \qquad (2.2.1.31)$$

With this, we have from (2.2.1.10) page 29 , (2.2.1.11) and (2.2.1.12) page 29:

$$xs = - \frac{1}{2} (p - \frac{4Z}{R}) x^2 \qquad (2.2.1.32)$$

$$n_\theta = - 1 + \frac{Z}{R} (p - \frac{4Z}{R}) x^2 \qquad (2.2.1.33)$$

$$m_\phi = 1 - \frac{1}{6h} (p - \frac{4Z}{R}) x^2 \qquad (2.2.1.34)$$

For the velocities, the solution of (2.2.1.17), (2.2.1.18) page 29 , without using the boundary condition at x = 1, yields:

$$\dot{w} = \dot{w}_o + Ax \qquad (2.2.1.35)$$

and:

$$\dot{v} = \frac{2Z}{R} x(\dot{w}_o + Ax) \qquad (2.2.1.36)$$

2. Solution for $u \leq x \leq 1$: regime 5-β

With $n_\phi = -1$, the equations of equilibrium (2.2.1.2) (2.2.1.3) page 28 still yield:

$$xs = -\frac{1}{2}\left(p - \frac{4Z}{R}\right)x^2 \qquad (2.2.1.32)$$

$$(0 \leq x \leq 1)$$

$$n_\theta = -1 + \frac{Z}{R}\left(p - \frac{4Z}{R}\right)x^2 \qquad (2.2.1.33)$$

$$(0 \leq x \leq 1)$$

From the relation corresponding to face β, which is

$$m_\theta - m_\phi = 1 \quad \text{(Table 2.1.3.1)}$$

and from the equation of equilibrium (2.2.1.1) page 28 we have:

$$m'_\phi = \frac{1}{x} - \frac{1}{2h}\left(p - \frac{4Z}{R}\right)x$$

Integrating between u and x, with $m_\phi(u^+) = 0$, we have:

$$m_\phi = \log\frac{x}{u} - \frac{1}{4h}\left(p - \frac{4Z}{R}\right)(x^2 - u^2) \qquad (2.2.1.35a)$$

then:

$$m_\theta = 1 + \log\frac{x}{u} - \frac{1}{4h}\left(p - \frac{4Z}{R}\right)(x^2 - u^2) \quad (2.2.1.36a)$$

For the velocities, we have the flow rules:

$\dot{e}_\theta = 0$, or with (2.1.1.1) page 28

$$\dot{v} = y'\dot{w} \qquad (2.2.1.37)$$

and $\dot{k}_\phi + \dot{k}_\theta = 0$ which becomes, with (2.1.1.2) and (2.1.1.4) page 19

$$\frac{(\dot{w} + y''\dot{v})'}{\dot{w}' + y''\dot{v}} = -\frac{1}{x}$$

and which gives:

$$\dot{w}' + y''\dot{v} = \frac{B}{x} \qquad (2.2.1.38)$$

Solving (2.2.1.37) and (2.2.2.38) for x, $\dot{w}$ and $\dot{v}$ and using shallow shell approximation, we have:

$$\dot{w} = B \log x \qquad (2.2.1.39)$$

$$\dot{v} = \frac{2Z}{R} x B \log x \qquad (2.2.1.40)$$

3. Boundary Matching

We have four unknowns: A, B, p, u. To determine them, we have four conditions:

(i) $m_\phi(u^-) = 0$

(ii) $m_\phi(1) = -1$

(iii) Continuity of $\dot{w}$ at $x = u$

(iiii) Continuity of $\dot{w}'$ at $x = u$

From $m_\phi(u^-) = 0$, and (2.2.1.34) page 39 , we have:

$$p = \frac{6h}{u^2} + \frac{4Z}{R} \qquad (2.2.1.41)$$

From $m(1) = -1$ and (2.2.1.35a) page 40, we have:

$$p = \frac{4h}{1-u^2}(1 - \log u) + \frac{4Z}{R} \qquad (2.2.1.42)$$

From $\Delta\dot{w} = 0$ and using (2.2.1.35) page 39 and (2.2.2.1.39) page 41, we have:

$$B \log u = Au + \dot{w}_o \qquad (2.2.1.43)$$

From $\Delta\dot{w}' = 0$, and differentiating (2.2.1.35) page 39 and (2.2.2.1.39) page 41, we have:

$$B/u = A \qquad (2.2.1.44)$$

Solving for A and B from (2.2.1.43) and (2.2.1.44), we have:

$$A = -\frac{\dot{w}_o}{u(1 - \log u)} \qquad (2.2.1.45)$$

and $$B = -\frac{\dot{w}_o}{1 - \log u} \qquad (2.2.1.46)$$

With these expressions of A and B, the velocities become:

From (2.2.1.35) and (2.2.1.36) page 39

$$\dot{w} = \dot{w}_o\left[1 - \frac{x}{u(1 - \log u)}\right] \qquad (0 \leq x \leq u) \qquad (2.2.1.47)$$

$$\dot{v} = \frac{2Z}{R}\dot{w}_o x\left[1 - \frac{x}{u(1 - \log u)}\right] \qquad (0 \leq x \leq u) \qquad (2.2.1.48)$$

From (2.2.1.39) and (2.2.1.40) page 41

$$\dot{w} = - \dot{w}_o \frac{\log x}{1 - \log u} \qquad (u \leq x \leq 1) \quad (2.2.1.49)$$

$$\dot{v} = - \frac{2Z}{R} \dot{w}_o \frac{x \log x}{1 - \log u} \quad (u \leq x \leq 1) \quad (2.2.1.50)$$

From (2.2.1.49), we have:

$$\dot{w}' = - \frac{\dot{w}_o}{x(1 - \log u)} \quad ,$$

and at $x = 1$, $\dot{w}' = - \dfrac{\dot{w}_o}{1 - \log u} \neq 0$, $\Delta\dot{w}' = \dfrac{\dot{w}_o}{1-\log u} > 0$

There is a hinge circle at $x = 1$, with $\dot{k}_\phi < 0$ as required

The value of u can be determined from (2.2.1.41) page 41 and (2.2.1.42) page 42 by eliminating p. Thus we have:

$$\frac{3}{u^2} = 5 - \log u^2 \qquad (2.2.1.51)$$

From this equation, we find:

$$u^2 \simeq 0.533 \quad \text{and}$$

$$u \simeq 0.731 \qquad (2.2.1.52)$$

Then p_s is given by either (2.2.1.41) page 41 or (2.2.1.42) page 42

$$p = p_s = \frac{4Z}{R} + \frac{6h}{u^2}$$

4. Admissibility

i. Kinematic Admissibility

a. $0 \leq x \leq u$, we must have $\dot{e}_\phi \leq 0$ and $\dot{k}_\theta \geq 0$.

We have: from (2.2.1.47) and (2.2.1.48):

$$\dot{e}_\phi = - \frac{(2Z/R)\dot{w}_o x}{u(1 - \log u)} < 0, \quad \text{since } \dot{w}_o > 0$$

and:

$$\dot{k}_\theta \simeq \frac{h}{x} \frac{\dot{w}_o}{u(1 - \log u)} > 0$$

b. $u \leq x \leq 1$, we must have $\dot{e}_\phi \leq 0$ and $\dot{k}_\theta \geq 0$.

Using now (2.2.1.49) and (2.2.1.50) page 43 we have:

$$\dot{e}_\phi = - \frac{2Z}{R} \frac{\dot{w}_o}{1 - \log u} < 0$$

and:

$$\dot{k}_\theta = - \frac{h}{x} (\dot{w}' + y''\dot{v}) \simeq \frac{h}{x^2} \frac{\dot{w}_o}{1 - \log u} > 0$$

When $\dot{k}_\theta$ is admissible, then $\dot{k}_\phi$ is also admissible. Thus the solution is kinematically admissible.

ii. Static Admissibility

a. n_θ, we must have $-1 \leq n_\theta \leq 0$.

From (2.2.1.33) page 39, and using p from (2.2.1.41) page 41, we have:

$$n_\theta = -1 + 6h \frac{Z}{R} \frac{x^2}{u^2} \qquad (2.2.1.53)$$

From this, we see that n_θ increases monotonically and at x = 1, we have:

$$n_\theta = -1 + \frac{6h}{u^2}\frac{Z}{R}$$

which is still negative since $h\frac{Z}{R}$ is very small compared to 1.

b. m_ϕ:

For $0 \leq x \leq u$, we must have $0 \leq m_\phi \leq 1$.

From (2.2.1.34) page 39 and using p from (2.2.1.41) page 41, we have:

$$m_\phi = 1 - \frac{x^2}{u^2} \quad (0 \leq x \leq u) \qquad (2.2.1.54)$$

From this, we can see that m_ϕ decreases monotonically from 1 to 0 as x increases from 0 to u. Thus m_ϕ is admissible in this interval.

For $u \leq x \leq 1$, we must have $-1 \leq m_\phi \leq 0$.

From (2.2.1.35a) page 40 and using p from (2.2.1.41) page 41, we have:

$$m_\phi = \log\frac{x}{u} - \frac{3}{2}\,\frac{x^2-u^2}{u^2} \quad , \; m_\phi(u) = 0$$

$$m_\phi' = \frac{1}{x} - \frac{3x}{u^2} = \frac{u^2-3x^2}{u^2 x} < 0 \qquad \text{since } u < x$$

Thus m_ϕ decreases monotonically from 0 to -1 as x increases from u to 1; m_ϕ is admissible in this interval.

The admissibility of m_θ follows, as a consequence of:

$$m_\theta = 1 \qquad (0 \le x \le u) \qquad \text{and}$$

$$m_\theta = 1 + m_\phi \quad (u \le x \le 1)$$

5. Conclusion

The solution is an exact one for the yield surface selected, and the static collapse pressure of a shallow spherical is given by (2.2.1.41), page 41

$$p = p_s = \frac{4Z}{R} + \frac{6h}{u^2}$$

with u^2 determined from (2.2.1.51) page 43, which is:

$$\frac{3}{u^2} = 5 - \log u^2$$

and which gives

$$u^2 \simeq 0.533 \quad \text{approximately.}$$

2.2.2 Dynamic Response of Shallow Spherical Shells

2.2.2.1 Low Pressure Range: $0 \leq p_o - p_s \leq 1.2h$

If the peak value of the dynamic pressure is sufficiently low, the whole shell will collapse in one regime: regime 5-7. (Fig. 4a,b)

1. Solution

VELOCITIES - The flow rules and the boundary conditions are the same as those in the corresponding static problem. From (2.2.1.20) and (2.2.1.21) page 31, we have:

$$\dot{w} = \dot{w}_o(1 - x) \qquad (2.2.1.20)$$

$$\dot{v} = \frac{2Z}{R}\,\dot{w}_o x(1 - x) \qquad (2.2.1.21)$$

$\dot{w}_o$ is the velocity at the apex $x = 0$ and is now a function of τ, and y' has the expression (2.2.1.30).

STRESSES - From the equilibrium equations (2.1.2.2), (2.1.2.3) page 21 and with $n_\phi = -1$, we have:

$$\begin{cases} -1 - y''xs - n_\theta = x\gamma\ddot{v} & (2.2.2.1.1) \\ -xy'' + (xs)' + y'n_\theta + px = x\gamma\ddot{w} & (2.2.2.1.2) \end{cases}$$

Moreover, from the flow rule $\dot{e}_\theta = 0$, we have:

$$\dot{v} = y'\dot{w} \quad \text{or}$$

$$\ddot{v} = y'\ddot{w} \qquad (2.2.2.1.3)$$

Solving (2.2.2.1.1), (2.2.2.1.2), (2.2.2.1.3), and with (2.2.1.20) page 31 we obtain:

$$xs = -\frac{1}{2}(p - \frac{4Z}{R})x^2 + \gamma\ddot{w}_o(\frac{x^2}{2} - \frac{x^3}{3}) \qquad (2.2.2.1.4)$$

and

$$n_\theta = -1 + \frac{2Z}{R}[\frac{1}{2}(p - \frac{4Z}{R})x^2 - \gamma\ddot{w}_o(\frac{3x^2}{2} - \frac{4x^3}{3})] \qquad (2.2.2.1.5)$$

From equation of equilibrium (2.1.2.1) page 21, with $m_\theta = 1$, and integrating from 0 to x, we have:

$$m_\phi = 1 - \frac{1}{6h}(p - \frac{4Z}{R})x^2 + \frac{\gamma\ddot{w}_o}{6h}(x^2 - \frac{x^3}{2}) \qquad (2.2.2.1.6)$$

From the boundary condition: $x = 1$, $m_\phi = 0$, we can determine $\gamma\ddot{w}_o$:

$$\gamma\ddot{w}_o = 2(p - p_s) \qquad (2.2.2.1.7)$$

Integrating (2.2.2.1.7) with respect to τ, where $p = p_o e^{-\tau}$ and using the initial condition: $\tau = 0$, $\dot{w}_o = 0$, we have:

$$\gamma\dot{w}_o = 2[p_o(1 - e^{-\tau}) - p_s\tau] \qquad (2.2.2.1.8)$$

FINAL RESULTS

With (2.2.2.1.7) and (2.2.2.1.8) above, we have

from (2.2.1.20) and (2.2.1.21) page 31

$$\gamma\dot{w} = 2[p_o(1 - e^{-\tau}) - p_s\tau](1-x) \qquad (2.2.2.1.9)$$

$$\gamma\dot{v} = \frac{4Z}{R}[p_o(1-e^{-\tau}) - p_s\tau]\, x(1-x) \qquad (2.2.2.1.10)$$

$$n_\phi = -1, \; m_\theta = 1$$

from (2.2.2.1.4) page 48, with:

$$\frac{4Z}{R} = p_s - 6h ,$$

we have:

$$xs = -\frac{6h - (p - p_s)}{2} x^2 - \frac{2(p - p_s)}{3} x^3 \quad (2.2.2.1.11)$$

from (2.2.2.1.5) page 48

$$n_\theta = -1 + \frac{2Z}{R}[\frac{6h - 5(p - p_s)}{2} x^2 + \frac{8}{3}(p - p_s)x^3]$$

(2.2.2.1.12)

from (2.2.2.1.6) page 48,

$$m_\phi = 1 - \frac{6h - (p - p_s)}{6h} x^2 - \frac{p - p_s}{6h} x^3 \quad (2.2.2.1.13)$$

2. Admissibility

i. Kinematic Admissibility

We must have $\dot{e}_\phi \leq 0$ and $\dot{k}_\theta \geq 0$. It has been seen previously (page 31) that these conditions are satisfied for $\dot{w}_o \geq 0$, or, from (2.2.2.1.8) page 48:

$$\gamma\dot{w}_o = 2[p_o(1 - e^{-\tau}) - p_s\tau] \geq 0$$

At $\tau = \tau_f$, where τ_f is determined by:

$$p_o(1 - e^{-\tau_f}) - p_s\tau_f = 0 \quad (2.2.2.1.14)$$

$\dot{w}_o = 0$, hence $\dot{w} = \dot{v} = 0$ and the motion ceases.

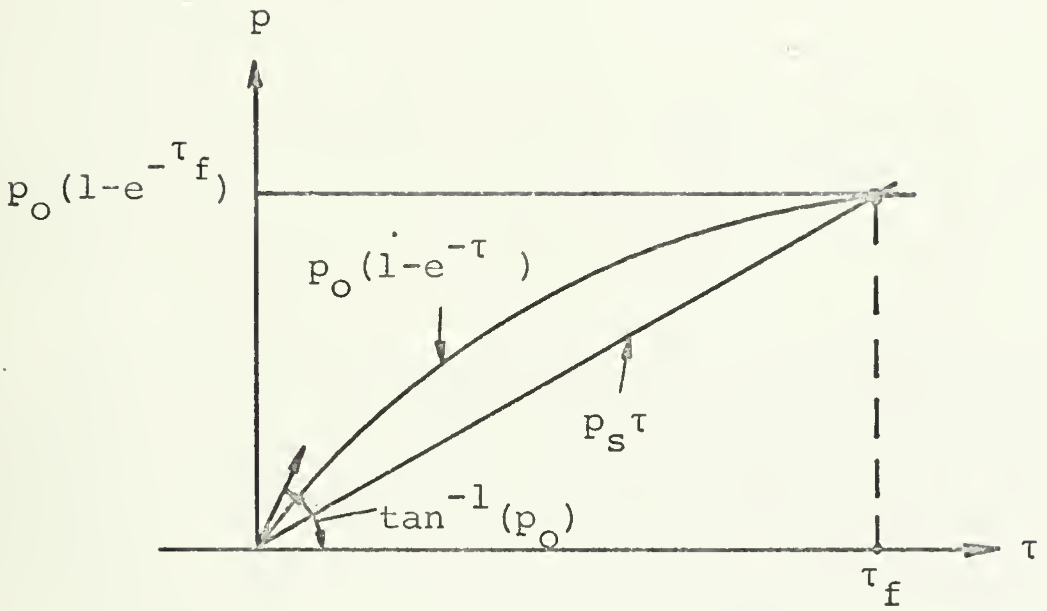


Fig. 8

Graphic Determination of τ_f

ii. Static Admissibility

a) m_ϕ, condition for admissibility: $0 \leq m_\phi \leq 1$

From (2.2.2.1.13), page 49, we derive:

$$m'_\phi = -\frac{1}{6h}\{\,2[6h-(p-p_s)]+3(p-p_s)x\,\}\,x \tag{2.2.2.1.15}$$

and

$$m''_\phi = -\frac{1}{6h}\{\,2[6h-(p-p_s)]+6(p-p_s)x\,\} \tag{2.2.2.1.16}$$

At $x = 0$,

$$m_\phi = 1 \text{ and } m'_\phi = 0$$

To be admissible, it is necessary that there be a maximum at x = 0 or

$$[m''_\phi]_{x=0} = \frac{(p - p_s) - 6h}{3h} < 0 \qquad \text{or}$$

$$p - p_s < 6h$$

Since $p = p_o e^{-\tau}$ decreases monotonically, the above inequality will be satisfied for any $\tau \geq 0$ if it is at $\tau = 0$ or:

$$p_o - p_s \leq 6h \qquad (2.2.2.1.17)$$

If $p_o - p_s = 6h$, then at $\tau = 0$, $(m''_\phi)_{x=0} = 0$, but $(m'''_\phi)_{x=0} = -\frac{p_o - p_s}{h} < 0$ and there is still a maximum at x = 0.

From (2.2.2.1.15) page 50 it can be seen that $m'_\phi = 0$ for another value x_m of x:

$$x_m = -\frac{2}{3}\,\frac{6h - (p - p_s)}{p - p_s} = \frac{2}{3}\,\frac{6h + (p_s - p)}{p_s - p} \qquad (2.2.2.1.18)$$

which is the position of the minimum of m_ϕ.

Assuming (2.2.2.1.17) above satisfied, then:

If $p - p_s > 0$, x_m is negative and m_ϕ decreases monotonically in the interval (0,1) from 1 to 0 and is therefore admissible.

If $p - p_s < 0$, x_m is positive. For m_ϕ to be admissible, x_m must be outside the interval (0,1), otherwise m_ϕ will

decrease in the interval $(0,x_m)$ from 1 to a minimum $(m_\phi)_{min}$ and increase from this minimum $(m_\phi)_{min}$ to zero in the interval $(x_m,1)$. Thus this minimum $(m_\phi)_{min}$, if it exists, will be negative, and make m_ϕ not admissible. Thus, we must have:

$$x_m \geq 1$$

or, from (2.2.2.1.18) page

$$12h + p_o e^{-\tau} - p_s \geq 0$$

Since $p_o e^{-\tau}$ decreases monotonically with τ, this inequality will be satisfied for any τ in the interval $[0,\tau_f]$, if it is at τ_f. Thus:

$$12h + p_o e^{-\tau_f} - p_s \geq 0 \qquad (2.2.2.1.19)$$

where τ_f has been determined in (2.2.2.1.14) page 49

b) n_θ, condition for admissibility: $-1 \leq n_\theta \leq 0$

From (2.2.2.1.12) page 49, we have:

$$n'_\theta = \frac{2Z}{R} x \{ [6h - 5(p - p_s)] + 8(p - p_s)x \} \qquad (2.2.2.1.20)$$

and $$(n'_\theta)_{x=0} = 0$$

Since $(n_\theta)_{x=0} = -1$, there must be a minimum at $x = 0$ or $(n''_\theta)_{x=0} \geq 0$ which requires

$$p_o e^{-\tau} - p_s \leq 1.2h$$

Since $p_o e^{-\tau}$ decreases monotonically with τ, this inequality will be satisfied for any $\tau \geq 0$ if it is at $\tau = 0$, or:

$$p_o - p_s \leq 1.2h \qquad (2.2.2.1.21)$$

From (2.2.2.1.20) page 52, we can see that $n'_\theta = 0$ for another value x_n of x:

$$x_n = -\frac{6h - 5(p - p_s)}{8(p - p_s)} = \frac{6h + 5(p_s - p)}{8(p_s - p)} \qquad (2.2.2.1.22)$$

which is the position of the maximum of n_θ. Assuming (2.2.2.1.21) page 53 satisfied, then:

If $p - p_s > 0$, x_n is negative and n_θ increases monotonically in the interval (0,1).

If $p - p_s < 0$, x_n is positive. If $x_n < 1$ or:

$$6h - 3(p_s - p) < 0 \qquad (2.2.2.1.23)$$

then n_θ has a maximum at x_n whose value, from (2.2.2.1.12) page 49, is:

$$(n_\theta)_{max} = -1 + \frac{1}{3}\frac{Z}{R}[6h + 5(p_s - p)]x_n^2$$

To be admissible, this maximum must be negative.

Since $x_n^2 \leq 1$, $(n_\theta)_{max}$ will be negative if

$$\frac{1}{3}\frac{Z}{R}[6h + 5(p_s - p)] \leq 1 \qquad \text{or:}$$

$$6h + 5(p_s - p) \leq 3\frac{R}{Z}$$

Since $p = p_o e^{-\tau}$ decreases monotonically with τ, this inequality will be satisfied for $0 \leq \tau \leq \tau_f$, if it is at $\tau = \tau_f$:

$$6h + 5(p_s - p_o e^{-\tau_f}) \leq 3\,\frac{R}{Z} \qquad (2.2.2.1.24)$$

From (2.2.2.1.19) page 52 we have:

$$p_s - p_o e^{-\tau_f} \leq 12h$$

Therefore:

$$6h + 5(p_s - p_o e^{-\tau_f}) \leq 6h + 5 \times 12h = 66h$$

and condition (2.2.2.1.24) above will be satisfied if we have:

$$66h \leq 3\frac{R}{Z} \qquad (2.2.2.1.25)$$

Since h is of the order of a few per cent, let us say 10%, and $\frac{Z}{R}$ is at most, about 1/3 for shallow shell, we find that, for these extreme values, we have:

$$66h = 6.6 \qquad \text{and}$$

$$3\frac{R}{Z} = 9$$

and

$$66h < 3\frac{R}{Z}$$

Thus, we can safely assume that condition (2.2.2.1.24) page 54 is always satisfied for thin, shallow shell. At $x = 1$, we must have

$$-1 \leq (n_\theta)_{x=1} \leq 0 \qquad \text{or:}$$

From (2.2.2.1.12) page 49, this condition is

$$-1 \leq -1 + 2\frac{Z}{R} \times \frac{18h + p - p_s}{6} \leq 0$$

For the first condition, we must have:

$$18h + p - p_s \geq 0$$

This condition will be satisfied for $0 \leq \tau \leq \tau_f$ if it is at $\tau = \tau_f$ or:

$$18h + p_o e^{-\tau_f} - p_s \geq 0 \qquad (2.2.2.1.26)$$

If we compare this condition with condition (2.2.2.1.19) page 52, we see that condition (2.2.2.1.26) will be satisfied if (2.2.2.1.19) is.

For the second condition, we must have:

$$18h + p - p_s \leq 3\frac{R}{Z}$$

This condition will be satisfied for $0 \leq \tau \leq \tau_f$ if it is at $\tau = 0$ or:

$$18h + p_o - p_s \leq 3\frac{R}{Z} \qquad (2.2.2.1.27)$$

From (2.2.2.1.21) page 53, which is: $p_o - p_s \leq 1.2h$, we have:

$$18h + p_o - p_s \leq 19.2h$$

and (2.2.2.1.27) will be satisfied if we have:

$$19.2h \leq 3\frac{R}{Z} \qquad (2.2.2.1.27)$$

Comparing this condition with (2.2.2.1.25) page 54 we can see that condition (2.2.2.1.27) is satisfied since (2.2.2.1.25) is.

iii. Conclusion

From the previous study, we have seen that:

-m_ϕ is admissible for:

$$p_o - p_s \le 6h \qquad (2.2.2.1.17) \text{ page } 51$$

$$12h + p_o e^{-\tau_f} - p_s \ge 0 \qquad (2.2.2.1.19) \text{ page } 52$$

-n_θ is admissible for:

$$p_o - p_s \le 1.2h \qquad (2.2.2.1.21) \text{ page } 53$$

$$18h + p_o e^{-\tau_f} - p_s \ge 0 \qquad (2.2.2.1.26) \text{ page } 55$$

We conclude therefore that the solution is statically admissible for:

$$p_o - p_s \le 1.2h \qquad (2.2.2.1.21)$$

and

$$12h + p_o e^{-\tau_f} - p_s \ge 0 \qquad (2.2.2.1.19)$$

3. Maximum Central Displacement

The displacement is maximum at $x = 0$. Its maximum is reached when $\tau = \tau_f$. Integrating (2.2.2.1.8) page 48, from 0 to τ_f with $\gamma w_o(\tau=0) = 0$ and using relation (2.2.2.1.14) page 49, we obtain:

$$\gamma w_o = 2[(p_o - p_s)\tau_f - \frac{1}{2} p_s \tau_f^2] \qquad (2.2.2.1.27a)$$

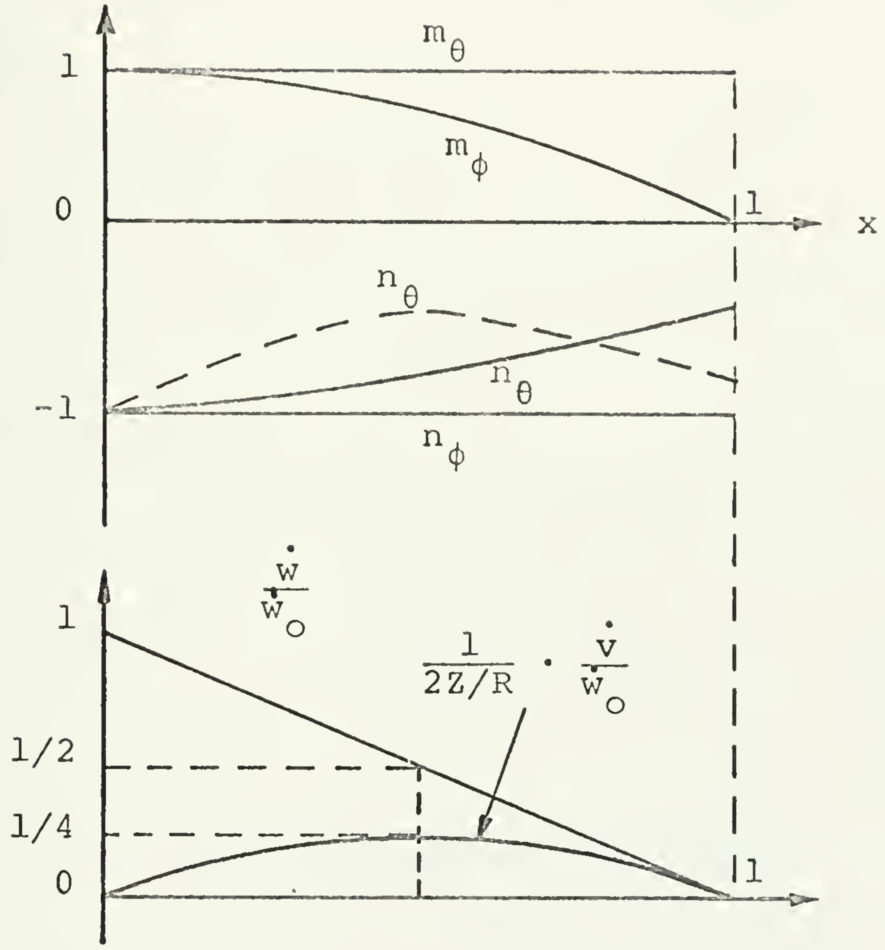


Fig. 9

Generalized Stress and Velocity Distributions

4. Energy Absorbed

The energy absorbed E_{abs} by the deformation of the shell is:

$$E_{abs} = \int_0^{t_f} \int_0^R P V_n \, d\sigma \, dt$$

V_n is the normal velocity: $V_n = R \frac{\dot{w}}{T_o}$

$d\sigma$ is the surface element: $d\sigma \simeq 2\pi r dr = 2\pi R^2 x dx$

P is the applied pressure: $P = \frac{N_o}{R} p$

t is the dimensional time: $t = T_o \tau$, and $dt = T_o d\tau$

Then:

$$E_{abs} = 2\pi R^2 N_o \int_0^{\tau_f} p d\tau \int_0^1 \dot{w} x dx$$

with $\dot{w}$ from (2.2.2.1.9) page 48, and $p = p_o e^{-\tau}$, we have:

$$E_{abs} = \frac{4\pi R^2 N_o}{\gamma} p_o \int_0^{\tau_f} [p_o(1 - e^{-\tau}) - p_s\tau]e^{-\tau}d\tau \int_0^1 (1 - x)xdx$$

$$E_{abs} = \frac{2\pi R^2 N_o}{3\gamma} \left\{ \frac{p_o^2}{2} (1 - e^{-\tau_f})^2 - p_o p_s [1 - (1+\tau_f)e^{-\tau_f}] \right\}$$

(2.2.2.1.28)

4. Final Displacement Components

With (2.2.2.1.27 a) page 56 and from (2.2.1.20) and (2.2.1.21) page 31, we have the following displacement distributions:

a) Normal Component:

$$\gamma w_f = 2[(p_o - p_s)\tau_f - \frac{1}{2} p_s \tau_f^2] \quad (1 - x)$$

(2.2.2.1.29)

b) Tangential Component:

$$\gamma v_f = 4\frac{Z}{R}[(p_o - p_s)\tau_f - \frac{1}{2} p_s \tau_f^2]x \ (1 - x)$$

(2.2.2.1.30)

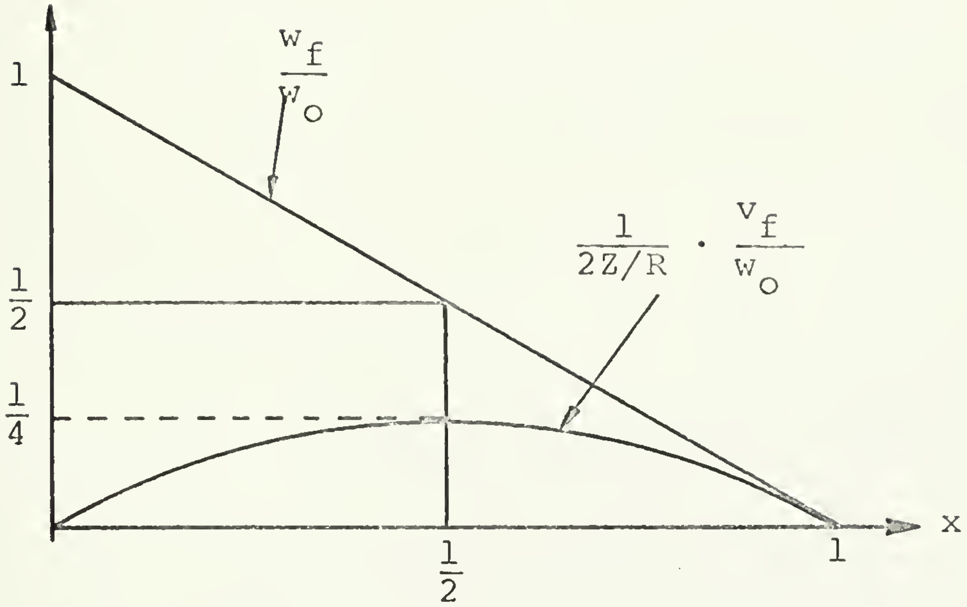


Fig. 10

Final Displacement Distributions

5. The Case of Rectangular Pressure Pulse

Let us consider a pressure pulse having the time variation:

$0 \leq \tau \leq 1 : p = p_o$

$\tau > 1 \quad : p = 0$

From (2.2.2.1.7) page 48 we have:

$$\gamma \ddot{w}_o = 2(p_o - p_s) \qquad (0 \leq \tau \leq 1)$$

$$\gamma \dot{w}_o = 2(p_o - p_s)\tau \qquad (0 \leq \tau \leq 1)$$

and:

$$\gamma \ddot{w}_o = -2p_s \qquad (\tau > 1)$$

$$\gamma \dot{w}_o = 2(p_o - p_s\tau) \qquad (\tau \geq 1)$$

Then we have:

$$\begin{cases} \gamma \dot{w} = 2(p_o - p_s)\tau\,(1 - x) & (0 \leq \tau \leq 1) \\ \gamma \dot{v} = \frac{4Z}{R}(p_o - p_s)\tau\, x(1 - x) & (0 \leq \tau \leq 1) \end{cases}$$

and:

$$\begin{cases} \gamma \dot{w} = 2(p_o - p_s\tau)\,(1 - x) & (\tau > 1) \\ \gamma \dot{v} = \frac{4Z}{R}(p_o - p_s\tau)\, x(1 - x) & (\tau > 1) \end{cases}$$

The solution is kinematically admissible for

$$0 \leq \tau \leq \tau_f \quad \text{where} \quad \tau_f = \frac{p_o}{p_s}$$

At $\tau = \tau_f$, the motion ceases.

The stresses are:

$$n_\phi = -1, \; m_\theta = 1$$

and from (2.2.2.1.12) and (2.2.2.1.13) page 49:

$$\begin{cases} n_\theta = -1 + \frac{2Z}{R}\left[\frac{6h - 5(p_o - p_s)}{2}x^2 + \frac{8}{3}(p_o - p_s)x^3\right] & (0 \leq \tau \leq 1) \\ m_\phi = -1 - \frac{6h - (p_o - p_s)}{6h}x^2 - \frac{p_o - p_s}{6h}x^3 & (0 \leq \tau \leq 1) \end{cases}$$

and

$$\begin{cases} n_\theta = -1 + \frac{2Z}{R}\left[\frac{6h + 5p_s}{2} x^2 - \frac{8}{3} p_s x^3\right] & (\tau > 1) \\ m_\phi = 1 - \frac{6h + p_s}{6h} x^2 + \frac{p_s}{6h} x^3 & (\tau > 1) \end{cases}$$

The static admissibility conditions are:

$$0 \leq p_o - p_s \leq 1.2h \quad (0 \leq \tau \leq 1)$$

which corresponds to (2.2.2.1.21) page 53 and

$$p_s \leq 12h$$

which corresponds to (2.2.2.1.19) page 52; with (2.2.1.28):

$$p_s = 6h + \frac{4Z}{R}$$

this latter condition is equivalent to:

$$\frac{4Z}{R} \leq 6h$$

or, with $h = H/2R = \frac{1}{4}\frac{2H}{R}$ (from 2.1.1.6):

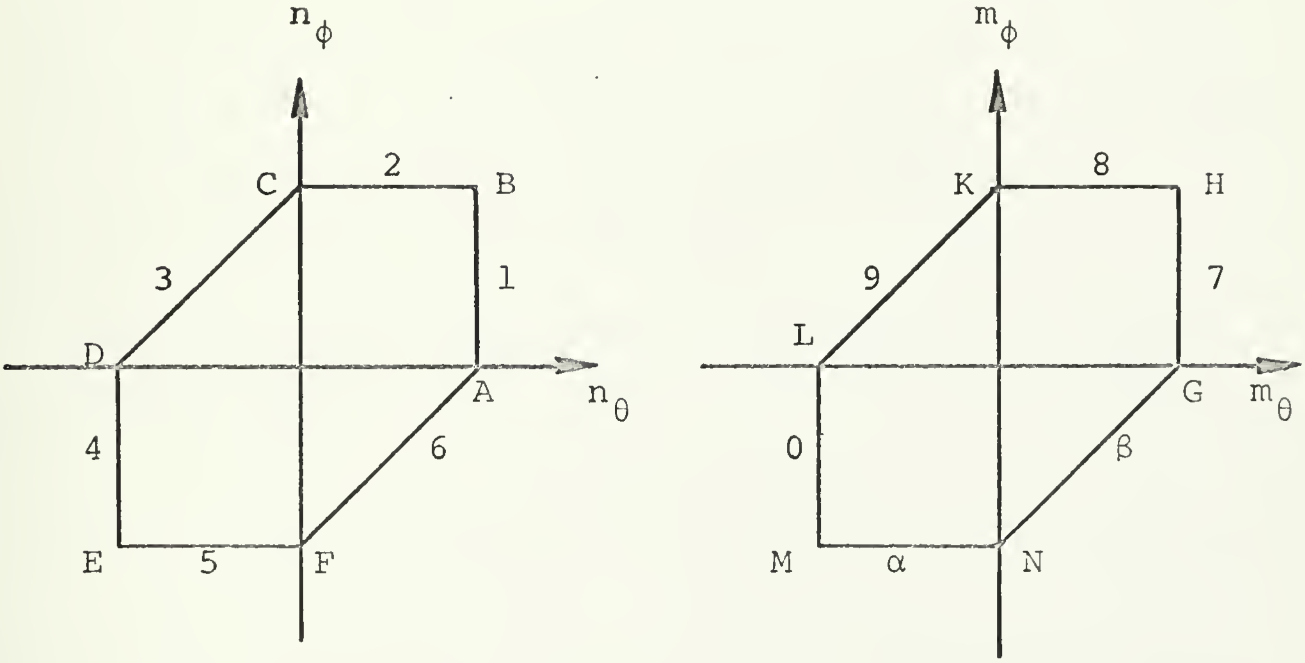
$$Z \leq \frac{3}{8}\,(2H)$$

Thus, for the solution to be admissible in the case of a rectangular pressure pulse, the height of the shell must be smaller than three-eighths of its thickness. This shell could therefore be considered as an imperfect plate.

2.2.2.2 Medium Pressure Range $1.2h \leq p_o - p_s \leq 6h$

A. First Stage of Motion: $0 \leq \tau \leq \tau_1$

When $p_o - p_s > 1.2h$, then from (2.2.2.1.21) page 53, we have $[n_\theta'']_{x=0} < 0$. Therefore n_θ would decrease from -1, which is not possible.



We first assume the following yield regimes:

$0 \leq x \leq u$: regime E - 7

$u \leq x \leq 1$: regime 5 - 7

The solution was found to be statically admissible, u is determined by:

$$u = \frac{15}{16} \frac{[p_o(1 - e^{-\tau}) - p_s\tau] - 1.2h\ \tau}{p_o(1 - e^{-\tau}) - p_s\tau}$$

and u decreases monotonically from its initial value u_o at $\tau = 0$:

$$u_o = \frac{15}{16}\,\frac{p_o - p_s - 1.2h}{p_o - p_s}$$

to $u = 0$ at $\tau = \tau_1$.

However, this solution is not kinematically admissible: while $\dot{e}_\theta$ remains negative for $0 \leq x \leq u$, $\dot{e}_\phi$ is negative only for $0 \leq x \leq 0.8u$ and becomes positive for $0.8u \leq x \leq u$, which is not admissible for regime E-7.

It turns out that the correct yield regimes for this stage of motion are:

$0 \leq x \leq u_1$: regime E-7

$u_1 \leq x \leq u_2$: regime 4-7

$u_2 \leq x \leq 1$: regime 5-7

The solutions are then:

1) $0 \leq x \leq u_1$: regime E-7

$$n_\phi = n_\theta = -1,\ \dot{e}_\phi \leq 0,\ \dot{e}_\theta \leq 0$$
$$m_\theta = 1, \qquad \dot{k}_\phi = -h(\dot{w}'+y''\dot{v})' = 0 \quad (2.2.2.2.1)$$

With $n_\phi = n_\theta = -1$, the equilibrium equations (2.1.2.2), (2.1.2.3) page 19 become:

$$\begin{cases} -y''\,xs = x\gamma\ddot{v} \\ (xs)' = xy'' + y' - px + 2\gamma\ddot{w} \end{cases}$$

For a spherical shallow shell, we have from (2.2.1.29)-(2.2.1.31) page 39

$$y = \frac{Z}{R}x^2, \quad y' = \frac{2Z}{R}x, \quad y'' = \frac{2Z}{R}$$

Then

$$\begin{cases} -\dfrac{2Z}{R}(xs) = x\gamma\ddot{v} & (2.2.2.2.2) \\ (xs)' = (\dfrac{4Z}{R} - p)x + x\gamma\ddot{w} & (2.2.2.2.3) \end{cases}$$

Let xs = Y, and solving the above equations for $\ddot{v}$ and $\ddot{w}$, we have:

$$\begin{cases} \gamma\ddot{v} = -\dfrac{2Z}{R}\cdot\dfrac{Y}{x} & (2.2.2.2.4) \\ \gamma\ddot{w} = \dfrac{Y'}{x} + p - \dfrac{4Z}{R} & (2.2.2.2.5) \end{cases}$$

From the flow rule $\dot{k}_\phi = 0$ in (2.2.2.2.1) page 63 we have:

$$\dot{w}' + y''\dot{v} = \frac{\dot{C}_o}{\gamma} \qquad (2.2.2.2.6)$$

where $\dot{C}_o$ is independent of x, but may be a function of τ. Differentiating (2.2.2.2.6) page 64 with respect to τ, we obtain:

$$\gamma\ddot{w}' + \frac{2Z}{R}\gamma\ddot{v} = \ddot{C}_o \qquad (2.2.2.2.7)$$

With $\gamma\ddot{v}$ from (2.2.2.2.4) page 64 and $\gamma\ddot{w}'$ by differentiating (2.2.2.2.5) page 64 with respect to x, equation (2.2.2.2.7) becomes:

$$x^2Y'' - xY' - 4\frac{Z^2}{R^2}\, x^2Y = \ddot{C}_o x^3 \qquad (2.2.2.2.8)$$

With $\xi = \frac{2Z}{R}x$, we obtain the following equation:

$$\xi^2\frac{d^2Y}{d\xi^2} - \xi\frac{dY}{d\xi} - \xi^2Y = \frac{\ddot{C}_o}{8\frac{Z^3}{R^3}}\,\xi^3 \qquad (2.2.2.2.9)$$

With the following conditions at $x = 0$:

at $x = \xi = 0$, $Y = 0$ and also, from (2.2.2.2.3) page 64 and since p and $\gamma\ddot{w}$ are finite:

at $x = \xi = 0 \quad Y' = 0$

the homogeneous equation of (2.2.2.2.9):

$$\xi^2\,\frac{d^2Y}{d\xi^2} - \xi\,\frac{dY}{d\xi} - \xi^2Y = 0$$

has a general solution Y_H expressible in terms of the modified Bessel functions of the first and second kind of order 1: I_1, and K_1 [19]

$$Y_H = \xi[C_1I_1(\xi) + C_2K_1(\xi)] \qquad (2.2.2.2.10)$$

The Wronskian of ξI_1, and ξK_1, is defined as:

$$W(\xi I_1, \xi K_1) = \begin{vmatrix} \xi I_1 & \xi K_1 \\ \frac{d}{d\xi}(\xi I_1) & \frac{d}{d\xi}(\xi K_1) \end{vmatrix}$$

Using the Abel's formula and using the approximate expressions of I_o, I_1, K_o, K_1 given in [19], where I_n, K_n are the modified Bessel functions of the first and second kind respectively, of order n, we have:

$$\xi(I_1K_o + I_oK_1) = 1 \qquad (2.2.2.2.11)$$

and

$$W(\xi I_1, \xi K_1) = -\xi \qquad (2.2.2.2.10a)$$

With this value of $W(\xi I_1, \xi K_1)$, and using the method of variation of parameters to solve equation (2.2.2.2.9) page 65 , we obtain:

$$Y = \frac{\ddot{C}_o}{8Z^3/R^3}\left[\xi I_1 \int_0^\xi \eta K_1(\eta)d\eta - \xi K_1 \int_0^\xi \eta I_1(\eta)\, d\eta\right]$$

$$+ \ddot{C}_1(\xi I_1) + \ddot{C}_2(\xi K_1)$$

Integrating by parts:

$$\int_0^\xi \eta K_1 d\eta = -\xi K_o + \int_0^\xi K_o(\eta)d\eta, \quad (\text{since } \frac{dK_o}{d\xi} = -K_1)$$

$$\int_0^\xi \eta I_1 d\eta = \xi I_o - \int_0^\xi I_o(\eta)d\eta, \quad (\text{since } \frac{dI_o}{d\xi} = I_1)$$

and noting that $\xi(I_1K_o + I_oK_1) = 1$ (2.2.2.2.11) we have:

$$Y = \frac{\ddot{C}_o}{8Z^3/R^3}\left[-\xi + \xi I_1 \int_0^\xi K_o d\eta + \xi K_1 \int_0^\xi I_o d\eta\right] + \ddot{C}_1(\xi I_1) + \ddot{C}_2(\xi K_1)$$

To determine $\ddot{C}_1$ and $\ddot{C}_2$, we have the initial conditions:

$$\xi = 0,\ Y = 0,\ Y' = 0$$

since $\lim\limits_{\xi \to 0} [\xi K_1(\xi)] = 1$

and $$\lim_{\xi \to 0} \int_0^\xi K_o d\eta \sim \lim_{\xi \to 0} \int_0^\xi -\log \eta d\eta$$

$$= \lim_{\xi \to 0} [-\eta \log \eta + \eta]_0^\xi = 0$$

we have:

$$Y(0) = 0 = \ddot{C}_2$$

Then:

$$\frac{dY}{d\xi} = \frac{\ddot{C}_o}{8Z^3/R^3} \left[-1 + \xi(I_1K_o + K_1I_o) + \xi I_o \int_0^\xi K_o d\eta - \xi K_o \int_0^\xi I_o d\eta \right] + \ddot{C}_1 \xi I_o$$

Or, with (2.2.2.2.11) page 66

$$\frac{dY}{d\xi} = \frac{\ddot{C}_o}{8t^3/R^3} \left[\xi I_o \int_0^\xi K_o d\eta - \xi K_o \int_0^\xi I_o d\eta \right] + \ddot{C}_1 \xi I_o$$

(2.2.2.2.12)

When $\xi = 0$, $\frac{dY}{d\xi} = 0$ for any finite value of $\ddot{C}_1$,

therefore:

$$Y = \frac{\ddot{C}_o}{8Z^3/R^3}\left[-\xi + \xi I_1 \int_0^{\xi} K_o d\eta + \xi K_1 \int_0^{\xi} I_o d\eta\right] + \ddot{C}_1 \xi I_1$$

$$(\xi = \frac{2Z}{R}x) \qquad (2.2.2.2.13)$$

From (2.2.2.2.4), page 64 we have:

$$\gamma\ddot{v} = -\frac{\ddot{C}_o}{2Z/R}\left[-1 + I_1 \int_0^{\xi} K_o d\eta + K_1 \int_0^{\xi} I_o d\eta\right] - \frac{4Z^2}{R^2}\ddot{C}_1 I_1$$

$$(2.2.2.2.14)$$

From (2.2.2.2.5), page 64 we have:

$$\gamma\ddot{w} = p - \frac{4Z}{R} + \frac{\ddot{C}_o}{2Z/R}\left[I_o \int_0^{\xi} K_o d\eta - K_o \int_0^{\xi} I_o d\eta\right] + \frac{4Z^2}{R^2}\ddot{C}_1 I_o$$

$$(2.2.2.2.15)$$

Integrating (2.2.2.2.14) and (2.2.2.2.15) page 68 with respect to τ with, at $\tau = 0$, $\dot{v} = \dot{w} = 0$, we have:

$$\gamma\dot{v} = -\frac{\dot{C}_o}{2Z/R}\left[-1 + I_1 \int_0^{\xi} K_o(\eta) d\eta + K_1 \int_0^{\xi} I_o(\eta) d\eta\right] - \frac{4Z^2}{R^2}\dot{C}_1 I_1$$

$$(0 \leq \tau \leq \tau_1),\ (0 \leq x \leq u_1),\ \xi = \frac{2Z}{R}x \qquad (2.2.2.2.16)$$

$$\gamma\dot{w} = p_o(1 - e^{-\tau}) - \frac{4Z}{R}\tau + \frac{\dot{C}_o}{2Z/R}\left[I_o \int_0^{\xi} K_o d\eta - K_o \int_0^{\xi} I_o d\eta\right.$$

$$\left. + \frac{4Z^2}{R^2}\dot{C}_1 I_o \right. \qquad (2.2.2.2.17)$$

$$(\xi = \frac{2Z}{R}x)$$

The initial conditions:

$$\dot{v}(\tau = 0) = \dot{w}(\tau = 0) = 0$$

will be satisfied if we take:

$$\dot{C}_o(0) = 0 \qquad (2.2.2.2.18)$$

$$\dot{C}_1(0) = 0 \qquad (2.2.2.2.19)$$

From the equilibrium equation (2.1.2.1) page 21 , with $m_\theta = 1$, we have

$$h(xm_\phi)' = h + Y$$

By integrating from 0 to ξ, with Y from (2.2.2.2.13) page 49, we have:

$$h\xi m_\phi = h\xi + \frac{\ddot{C}_o}{8Z^3/R^3}\left\{ -\frac{\xi^2}{2} + \int_0^\xi \eta I_1(\eta)\left[\int_0^\eta K_o(\eta_1)d\eta_1\right]d\eta \right.$$

$$\left. + \int_0^\xi \eta K_1(\eta)\left[\int_0^\eta I_o(\eta_1)d\eta_1\right]d\eta \right\} + \ddot{C}_1\int_0^\xi \eta I_1(\eta)d\eta,$$

or, after some integrations by parts:

$$h\xi m_\phi = h\xi + \frac{\ddot{C}_o}{8Z^3/R^3}\left\{ -\frac{\xi^2}{2} + \xi I_o\int_0^\xi K_o d\eta - \xi K_o\int_0^\xi I_o d\eta \right.$$

$$+ \int_0^\xi K_o(\eta)\left[\int_0^\eta I_o(\eta_1)d\eta_1\right]d\eta \qquad (2.2.2.2.20)$$

$$\left(\xi = \frac{2Z}{R}x\right)$$

$$\left. - \int_0^\xi I_o(\eta)\left[\int_0^\eta K_o(\eta_1)d\eta_1\right]d\eta \right\} + \ddot{C}_1\int_0^\xi \eta I_1 d\eta$$

Shallow Shell Approximation

With $y'^2 = (\frac{2Z}{R} x)^2 = \xi^2 << 1$, we have the following limited series expansions:

$$I_o(\xi) \simeq 1 + \frac{\xi^2}{4}$$

$$K_o(\xi) \simeq -(\gamma_E + \log \frac{\xi}{2})(1 + \frac{\xi^2}{4}) + \frac{\xi^2}{4}$$

Where γ_E is the Euler's constant

$$\gamma_E = \lim_{n\to\infty} \left[\left(\sum_{p=1}^{n} \frac{1}{p} \right) - \log n \right] \simeq 0.5772$$

$$I_1(\xi) = \frac{\xi}{2} + \frac{\xi^3}{16}$$

$$K_1(\xi) = \frac{1}{\xi} + \frac{\xi}{4} - \frac{\xi}{2}(1 - \gamma_E - \log \frac{\xi}{2}) - \frac{\xi^3}{16}(1 - \gamma_E - \log \frac{\xi}{2})$$

$$\int_o^\xi K_o(\eta)d\eta \simeq \xi(1 - \gamma_E - \log \frac{\xi}{2}) + \frac{\xi^3}{12}(\frac{4}{3} - \gamma_E - \log \frac{\xi}{2})$$

$$\int_o^\xi I_o(\eta)d\eta \simeq \xi + \frac{1}{12}\xi^3$$

With these results, we obtain:

$$Y = xs = x^2 \left[\frac{2Z^2}{R^2} \ddot{C}_1 + \frac{1}{3} \ddot{C}_o x \right] \qquad (2.2.2.2.21)$$

$$\gamma\dot{v} = -\frac{2Z}{R} x \left(\frac{2Z^2}{R^2} \dot{C}_1 + \frac{1}{3} \dot{C}_o x\right) \qquad (2.2.2.2.22)$$

$$\gamma\dot{w} = p_o(1 - e^{-\tau}) - \frac{4Z}{R}\tau + \frac{4Z^2}{R^2}\dot{C}_1 + \dot{C}_o x$$

(2.2.2.2.23)

$$hxm_\phi = hx + \frac{2}{3}\frac{Z^2}{R^2}\ddot{C}_1 x^3 + \frac{1}{12}\ddot{C}_o x^4 \qquad (2.2.2.2.24)$$

2) $u_1 \leq x \leq u_2$: regime 4-7

For this regime, we have $n_\theta = -1$, $m_\theta = 1$ and the flow rules are:

$$\dot{e}_\phi = 0 \rightarrow \dot{v}' - y''\dot{w} = 0 \qquad (2.2.2.2.25)$$

$$\dot{k}_\phi = 0 \rightarrow \dot{w}' + y''\dot{v} = \dot{A}/\gamma \qquad (2.2.2.2.26)$$

where $\dot{A}$ is independent of x, and may be a function of τ.

Eliminating $\dot{w}$, we obtain, with $y'' = 2Z/R$ from (2.2.1.31):

$$(\gamma\dot{v})'' + (\frac{2Z}{R})^2(\gamma\dot{v}) = \frac{2Z}{R}\dot{A} \qquad (2.2.2.2.27)$$

The general solution of (2.2.2.2.27) is:

$$\gamma\dot{v} = \frac{\dot{A}}{2Z/R} + \dot{B}\cos\frac{2Z}{R}x + \dot{C}\sin\frac{2Z}{R}x \qquad (2.2.2.2.28)$$

Then, from (2.2.2.2.25), page 70

$$\gamma\dot{w} = \dot{C}\cos\frac{2Z}{R}x - \dot{B}\sin\frac{2Z}{R}x \qquad (2.2.2.2.29)$$

$\dot{B}$, $\dot{C}$ are constants of integration which may be functions of τ.

The equations of motion (2.1.2.2), (2.1.2.3) become, with $n_\theta = -1$ and $y'' = 2Z/R$:

$$(xn_\phi)' - \frac{2Z}{R}(xs) + 1 = x\gamma\ddot{v} \qquad (2.2.2.2.30)$$

$$\frac{2Z}{R}(xn_\phi) + (xs)' + (p - \frac{2Z}{R})x = x\gamma\ddot{w} \qquad (2.2.2.2.31)$$

Elimination of xs from (2.2.2.2.30) and (2.2.2.2.31) yields:

$$(xn_\phi)'' + (\frac{2Z}{R})^2(xn_\phi) = -\frac{2Z}{R}(p - \frac{2Z}{R})x + \gamma\ddot{v} + x[(\gamma\ddot{v})' + \frac{2Z}{R}\gamma\ddot{w}]$$

Having $\gamma\ddot{v}$ and $\gamma\ddot{w}$ by differentiating (2.2.2.2.28) and (2.2.2.2.29) with respect to τ, we have:

$$(xn_\phi)'' + (\frac{2Z}{R})^2(xn_\phi) = -\frac{2Z}{R}(p - \frac{2Z}{R})x + \frac{\ddot{A}}{2Z/R} + \ddot{B}\cos\frac{2Z}{R}x + \ddot{C}\sin\frac{2Z}{R}x + \frac{4Z}{R}x(\ddot{C}\cos\frac{2Z}{R}x - \ddot{B}\sin\frac{2Z}{R}x) \qquad (2.2.2.2.32)$$

The general solution of (2.2.2.2.28) is:

$$xn_\phi = \ddot{D}\cos\frac{2Z}{R}x + \ddot{E}\sin\frac{2Z}{R}x + \frac{\ddot{A}}{8Z^3/R^3} - \frac{p - 2Z/R}{2Z/R}x + \frac{\ddot{B}}{2}x^2\cos\frac{2Z}{R}x + \frac{\ddot{C}}{2}x^2\sin\frac{2Z}{R}x \qquad (2.2.2.2.33)$$

$\ddot{D}$ and $\ddot{E}$ are constants of integration which may be a function of τ.

From equation (2.2.2.2.30) we have then:

$$xs = -\frac{\ddot{Q}}{4Z^2/R^2} - \frac{\ddot{A}}{4Z^2/R^2}x + \ddot{E}\cos\frac{2Z}{R}x - \ddot{D}\sin\frac{2Z}{R}x + \frac{\ddot{C}}{2}x^2\cos\frac{2Z}{R}x - \frac{\ddot{B}}{2}x^2\sin\frac{2Z}{R}x \qquad (2.2.2.2.34)$$

where $\ddot{Q} = p - \frac{4Z}{R}$, or with p_s from (2.2.1.28):

$$\ddot{Q} = p - p_s + 6h \qquad (2.2.2.2.35)$$

From the equilibrium equation (2.1.2.1)

with $m_\theta = 1$, we have:

$$h(xm_\theta)' = h + xs \qquad \text{and}$$

$$h(xm_\phi) = \ddot{F} + hx - \frac{\ddot{Q}}{4Z^2/R^2}x - \frac{\ddot{A}}{8Z^2/R^2}x^2 + \frac{\ddot{E}}{2Z/R}\sin\frac{2Z}{R}x$$

$$+ \frac{\ddot{D}}{2Z/R}\cos\frac{2Z}{R}x + \frac{\ddot{C}}{2}\left[\frac{2x}{4Z^2/R^2}\cos\frac{2Z}{R}x\right.$$

$$\left.+ \frac{(4Z^2/R^2)x^2 - 2}{8Z^3/R^3}\sin\frac{2Z}{R}x\right]$$

$$- \frac{\ddot{B}}{2}\left[\frac{2x}{4Z^2/R^2}\sin\frac{2Z}{R}x - \frac{(4Z^2/R^2)x^2 - 2}{8Z^3/R^3}\cos\frac{2Z}{R}x\right]$$

(2.2.2.2.36)

Shallow shell approximation

Using the shallow shell approximation $y'^2 = \frac{4Z^2}{R^2}x^2 << 1$, we have the following series expansions:

$$\cos\frac{2Z}{R}x = 1 - \frac{2Z^2}{R^2}x^2 + \frac{2}{3}\frac{Z^4}{R^4}x^4 - \frac{8}{90}\frac{Z^6}{R^6}x^6$$

$$\sin\frac{2Z}{R}x = \frac{2Z}{R}x - \frac{4}{3}\frac{Z^3}{R^3}x^3 + \frac{4}{15}\frac{Z^5}{R^5}x^5$$

With these results, we obtain:

$$\gamma\dot{v} = \frac{\dot{A} + \frac{2Z}{R}\dot{B}}{2Z/R} + \frac{2Z}{R}x(\dot{C} - \frac{Z}{R}\dot{B}x) \qquad (2.2.2.2.37)$$

$$\gamma\dot{w} = \dot{C} - \frac{2Z}{R}\dot{B}x \qquad (2.2.2.2.38)$$

$$xn_\phi = \frac{\ddot{A} + \frac{8Z^3}{R^3}\ddot{D}}{8Z^3/R^3} + \frac{\frac{4Z^2}{R^2}\ddot{E} + \frac{2Z}{R} - p}{2Z/R}\,x + \frac{1}{2}\,\frac{\frac{2Z}{R}\ddot{B} - \frac{8Z^3}{R^3}\ddot{D}}{2Z/R}\,x^2$$

$$+ \frac{1}{3}\frac{Z}{R}\left(3\ddot{C} - \frac{4Z^2}{R^2}\ddot{E}\right)x^3 + \frac{1}{12}\frac{Z}{R}\left(8\frac{Z^3}{R^3}\ddot{D} - 12\frac{Z}{R}\ddot{B}\right)x^4$$

$$(2.2.2.2.39)$$

$$xs = \frac{\frac{4Z^2}{R^2}\ddot{E} - \ddot{Q}}{4Z^2/R^2} - \frac{\ddot{A} + \frac{8Z^3}{R^3}\ddot{D}}{4Z^2/R^2}\,x + \frac{1}{2}\left(\ddot{C} - \frac{4Z^2}{R^2}\ddot{E}\right)x^2$$

$$+ \frac{1}{6}\left(8\frac{Z^3}{R^3}\ddot{D} - 6\frac{Z}{R}\ddot{B}\right)x^3 \qquad (2.2.2.2.40)$$

$$hxm_\phi = \frac{16\frac{Z^4}{R^4}\ddot{F} + 8\frac{Z^3}{R^3}\ddot{D} - \frac{2Z}{R}\ddot{B}}{16Z^4/R^4} + \left[h + \frac{\frac{4Z^2}{R^2}\ddot{E} - \ddot{Q}}{4Z^2/R^2}\right]x - \frac{\ddot{A} + \frac{8Z^3}{R^3}\ddot{D}}{8Z^2/R^2}\,x^2$$

$$+ \frac{1}{6}\left(\ddot{C} - \frac{4Z^2}{R^2}\ddot{E}\right)x^3 + \frac{1}{24}\left(8\frac{Z^3}{R^3}\ddot{D} - 6\frac{Z}{R}\ddot{B}\right)x^4$$

$$(2.2.2.2.41)$$

3) $u_2 \leq x \leq 1$: regime 5-7

For this regime, we have:

$$n_\phi = -1 \quad , \quad m_\theta = 1 \, ,$$

and the flow rules are:

$$\dot{e}_\theta = 0 \text{ and } \dot{k}_\phi = 0$$

With the solutions from the low pressure range, and observing that in this case, we cannot use the boundary

conditions at $x = 0$, we obtain the following results:

$$\gamma\dot{v} = \frac{2Z}{R}\dot{K}x\,(x - 1) \qquad (2.2.2.2.42)$$

$$\gamma\dot{w} = \dot{K}\,(x - 1) \qquad (2.2.2.2.43)$$

$$n_\theta = -1 - \frac{2Z}{R}\left[\ddot{G} - h - \frac{3\ddot{K} + \ddot{Q}}{2}x^2 + \frac{4}{3}\ddot{K}x^3\right]$$

$$(2.2.2.2.44)$$

$$xs = \ddot{G} - h - \frac{\ddot{K} + \ddot{Q}}{2}x^2 + \frac{1}{3}\ddot{K}x^3 \qquad (2.2.2.2.45)$$

$$hxm_\phi = \frac{1}{12}\ddot{K} + \frac{1}{6}\ddot{Q} - \ddot{G} + \ddot{G}x - \frac{\ddot{K} + \ddot{Q}}{6}x^3 + \frac{1}{12}\ddot{K}x^4$$

$$(2.2.2.2.46)$$

$\ddot{G}$, $\dot{K}$ and its derivative $\ddot{K}$ are constants of integration which may be functions of τ, $\ddot{Q}$ is known and has been defined in (2.2.2.2.35).

4) Summary

Let us integrate (2.2.2.2.35) with $\dot{Q}(0) = 0$:

$$\dot{Q} = p_o(1 - e^{-\tau}) - \frac{4Z}{R}\tau = p_o(1 - e^{-\tau}) - (p_s - 6h)\tau$$

$$(2.2.2.2.47)$$

With this notation we have the following results:

a) $0 \leq x \leq u_1$: regime E -7

$$n_\phi = n_\theta = -1 \quad , \quad m_\theta = 1$$

and from (2.2.2.2.21)-(2.2.2.2.24):

$$xs = x^2 [\frac{2Z^2}{R^2} \ddot{C}_1 + \frac{1}{3} \ddot{C}_o x] \tag{2.2.2.2.21}$$

$$hxm_\phi = hx + \frac{2}{3} \frac{Z^2}{R^2} \ddot{C}_1 x^3 + \frac{1}{12} \ddot{C}_o x^4 \tag{2.2.2.2.24}$$

$$\gamma \dot{w} = \dot{Q} + \frac{4Z^2}{R^2} \dot{C}_1 + \dot{C}_o x \tag{2.2.2.2.23}$$

$$\gamma \dot{v} = - \frac{2Z}{R} x (\frac{2Z^2}{R^2} \dot{C}_1 + \frac{1}{3} \dot{C}_o x) \tag{2.2.2.2.22}$$

b) $u_1 \le x \le u_2$: regime 4-7

$$n_\theta = -1 \quad , \quad m_\theta = 1$$

and from (2.2.2.2.37)-(2.2.2.2.41)

$$xn_\phi = \frac{\ddot{A} + \frac{8Z^3}{R^3} \ddot{D}}{8Z^3/R^3} + \frac{\frac{4Z^2}{R^2} \ddot{E} + \frac{2Z}{R} - p}{2Z/R} x + \frac{1}{2} \frac{\frac{2Z}{R} \ddot{B} - \frac{8Z^3}{R^3} \ddot{D}}{2Z/R} x^2$$

$$+ \frac{1}{3} \frac{Z}{R} (3\ddot{C} - \frac{4Z^2}{R^2} \ddot{E}) x^3 + \frac{1}{12} \frac{Z}{R} (\frac{8Z^3}{R^3} \ddot{D} - 12 \frac{Z}{R} \ddot{B}) x^4 \tag{2.2.2.2.39}$$

$$xs = \frac{\frac{4Z^2}{R^2} \ddot{E} - \ddot{Q}}{4Z^2/R^2} - \frac{\ddot{A} + \frac{8Z^3}{R^3} \ddot{D}}{4Z^2/R^2} x + \frac{1}{2} (\ddot{C} - \frac{4Z^2}{R} \ddot{E}) x^2$$

$$+ \frac{1}{6} (\frac{8Z^3}{R^3} \ddot{D} - 6 \frac{Z}{R} \ddot{B}) x^3 \tag{2.2.2.2.40}$$

$$hxm_\phi = \frac{16\frac{Z^4}{R^4}\ddot{F} + 8\frac{Z^3}{R^3}\ddot{D} - \frac{2Z}{R}\ddot{B}}{16Z^4/R^4} + [h + \frac{\frac{4Z^2}{R^2}\ddot{E} - \ddot{Q}}{4Z^2/R^2}] x - \frac{\ddot{A} + \frac{8Z^3}{R^3}\ddot{D}}{8Z^2/R^2} x^2$$

$$+ \frac{1}{6} (\ddot{C} - \frac{4Z^2}{R^2} \ddot{E}) x^3 + \frac{1}{24} (8 \frac{Z^3}{R^3} \ddot{D} - 6 \frac{Z}{R} \ddot{B}) x^4 \tag{2.2.2.2.41}$$

$$\gamma\dot{w} = \dot{C} - \frac{2Z}{R}\dot{B}x \qquad (2.2.2.2.38)$$

$$\gamma\dot{v} = \frac{\dot{A} + \frac{2Z}{R}\dot{B}}{2Z/R} + \frac{2Z}{R}\,x\,(\dot{C} - \frac{Z}{R}Bx) \qquad (2.2.2.2.37)$$

c) $u_2 \le x \le 1$: regime 5-7

$$n_\phi = -1 \quad , \quad m_\theta = 1$$

and from (2.2.2.2.42)-(2.2.2.2.46), we have:

$$n_\theta = -1 - \frac{2Z}{R}\,[\ddot{G} - h - \frac{3\ddot{K} + \ddot{Q}}{2}x^2 + \frac{4}{3}\ddot{K}x^3] \qquad (2.2.2.2.44)$$

$$xs = \ddot{G} - h - \frac{\ddot{K} + \ddot{Q}}{2}x^2 + \frac{1}{3}\ddot{K}x^3 \qquad (2.2.2.2.45)$$

$$hxm_\phi = \frac{1}{12}\ddot{K} + \frac{1}{6}\ddot{Q} - \ddot{G} + \ddot{G}x - \frac{\ddot{K} + \ddot{Q}}{6}x^3 + \frac{1}{12}\ddot{K}x^4$$

$$(2.2.2.2.46)$$

$$\gamma\dot{v} = \frac{2Z}{R}\dot{K}x\,(x - 1) \qquad (2.2.2.2.42)$$

$$\gamma\dot{w} = \dot{K}\,(x - 1) \qquad (2.2.2.2.43)$$

5) Boundary Matching

There are 12 unknowns: $\dot{C}_o$, $\dot{C}_1$, $\dot{A}$, $\frac{2Z}{R}\dot{B}$, $\dot{C}$, $\frac{8Z^3}{R^3}\ddot{D}$, $\frac{4Z^2}{R^2}\ddot{E}$, $\frac{16Z^4}{R^4}\ddot{F}$, $\dot{K}$, $\ddot{G}$, u_1, u_2,

To determine these unknowns, we have the following conditions of continuity:

a) Continuity of n_ϕ, s, m_ϕ from equilibrium considerations [17].

b) Continuity of $\gamma\dot{w}$ because of the assumption that shear

strains are negligible [17], and also from the fact that $\gamma\dot{w}$ must be continuous in τ, otherwise the acceleration would be infinite, which is not possible because the load applied is finite. Since $\gamma\dot{w}$ is a function of u_1 and u_2, and u_1 and u_2 are functions of τ, it is necessary that $\gamma\dot{w}$ be continuous across $x = u_1$ and $x = u_2$.

c) Continuity of $\gamma\dot{v}$: Here $\gamma\dot{v}$ must be continuous with respect to τ and thus must be continuous across $x = u_1$ and $x = u_2$.

d) Continuity of $\gamma\dot{w}'$. $\gamma\dot{w}'$ may be discontinuous. From the yield surface and the direction of the strain rate vector, this discontinuity can occur if $|m_\phi| = 1$, which is not the case here since at $x = u_1$ and $x = u_2$, $|m_\phi| < 1$. Therefore $\gamma\dot{w}'$ must be continuous across $x = u_1$ and $x = u_2$.

Thus we have six conditions of continuity at each boundary resulting in 12 equations to determine the 12 unknowns listed above.

Solving and using the shallow shell approximations whenever possible, we have (Appendix A1):

$$\ddot{C} = 2(p - p_s) \qquad (2.2.2.2.48)$$

$$\dot{C} = 2[p_o(1 - e^{-\tau}) - p_s\tau] \qquad (2.2.2.2.49)$$

$$\dot{K} = -2[p_o(1 - e^{-\tau}) - p_s\tau] \qquad (2.2.2.2.50)$$

$$\frac{2Z}{R}\dot{B} = 2[p_o(1 - e^{-\tau}) - p_s\tau] \qquad (2.2.2.2.51)$$

$$\dot{C}_o = -2[p_o(1 - e^{-\tau}) - p_s\tau] \qquad (2.2.2.2.52)$$

$$\frac{4Z^2}{R^2}\dot{C}_1 = p_o(1 - e^{-\tau}) - p_s\tau - 6h\tau \qquad (2.2.2.2.53)$$

$$\dot{A} = -\dot{C}(1 + \frac{2Z^2}{R^2}u_2{}^2) = -2[p_o(1 - e^{-\tau}) - p_s\tau](1 + \frac{2Z^2}{R^2}u_2{}^2)$$

$$(2.2.2.2.54)$$

$$\ddot{D}_1 = \frac{8Z^3}{R^3}\ddot{D} - \ddot{C} = \frac{2Z^2}{R^2}(\ddot{C}u_2{}^2 + 2\dot{C}\dot{u}_2u_2) + 0(\frac{Z^4}{R^4})$$

$$(2.2.2.2.55)$$

$$\ddot{E}_1 = \frac{4Z^2}{R^2}\ddot{E} - \ddot{Q} = \frac{2}{3}\frac{Z^4}{R^4}[6(\ddot{C}u_2^2 + 2\dot{C}\dot{u}_2u_2)u_1^2 - 4(3\ddot{C} - \ddot{Q})u_1^3 + 5\ddot{C}u_1^4]u_1$$

$$(2.2.2.2.56)$$

$$\frac{\ddot{F}_1}{4Z^2/R^2} = \frac{\frac{16Z^4}{R^4}\ddot{F} + \frac{8Z^3}{R^3}\ddot{D} - \ddot{C}}{4Z^2/R^2} = -\frac{1}{3}\frac{Z^4}{R^4}[6(\ddot{C}u_2^2 + 2\dot{C}\dot{u}_2u_2)u_1^2$$

$$- 4(3\ddot{C} - \ddot{Q})u_1^3 + 5\ddot{C}u_1^4]u_1^2 \qquad (2.2.2.2.57)$$

$$\ddot{G} = h + 0(\frac{Z^2}{R^2}) \qquad (2.2.2.2.58)$$

u_1 and u_2 are to be determined from:

$$u_2{}^2 = \frac{3\dot{C} - \dot{Q}}{\dot{C}}u_1 - \frac{5}{3}u_1{}^2 \qquad (2.2.2.2.59)$$

$$6(\ddot{C}u_2{}^2 + 2\dot{C}\dot{u}_2u_2)(u_1 + u_2) - 4(3\ddot{C} - \ddot{Q})(u_1{}^2 + u_1u_2 + u_2{}^2)$$

$$+ 5\ddot{C}(u_1{}^2 + u_2{}^2)(u_1 + u_2) = 0 \qquad (2.2.2.2.60)$$

This system is equivalent to a first order ordinary differential equation. Being highly non-linear, it has to be solved numerically. (Sample output in Appendix A3)

6) Final Results

With the unknowns thus determined, in terms of u_1 and u_2, we have the following results:

a) $0 \leq x \leq u_1$: regime E-7 (Figs. 4a,b)

$$n_\theta = n_\phi = -1$$

$$m_\theta = 1$$

$$s = -\frac{6h - (p - p_s)}{2} x - \frac{2}{3} (p - p_s) x^2 \quad (2.2.2.2.61)$$

$$m_\phi = 1 - \frac{6h - (p - p_s)}{6h} x^2 - \frac{p - p_s}{6h} x^3 \quad (2.2.2.2.62)$$

$$\gamma \dot{w} = 2[p_o(1 - e^{-\tau}) - p_s\tau](1 - x) \quad (2.2.2.2.63)$$

$$\gamma \dot{v} = \frac{2Z}{R} x \left\{ \frac{6h\tau - [p_o(1-e^{-\tau}) - p_s\tau]}{2} + \frac{2[p_o(1-e^{-\tau}) - p_s\tau]}{3} x \right\} \quad (2.2.2.2.64)$$

b) $u_1 \leq x \leq u_2$: regime 4-7 (Figs. 4a,b)

$$n_\theta = -1 \quad , \quad m_\theta = 1$$

$$n_\phi = -1 + \frac{Z}{R} \frac{1}{x} \left\{ \frac{5(p-p_s) - 6h}{3} \left[x^3 - \frac{u_1^2 + u_1 u_2 + u_2^2}{u_1 + u_2} x^2 + \frac{u_1^2 u_2^2}{u_1 + u_2} \right] - \frac{5(p - p_s)}{6} [x^4 - (u_1^2 + u_2^2) x^2 + u_1^2 u_2^2] \right\} \quad (2.2.2.2.65a)$$

After some manipulations, n_ϕ can be written in the form:

$$n_\phi = -1 + \frac{5}{6}\frac{Z}{R}\frac{(x-u_1)(u_2-x)}{x}\Big\{ (p-p_s)\, x^2 - [2(p-p_s - 1.2h)$$

$$- (p-p_s)(u_1+u_2)]\, x - [2(p-p_s - 1.2h)$$

$$- (p-p_s)(u_1+u_2)]\, \frac{u_1u_2}{u_1+u_2} \Big\} \qquad (2.2.2.2.65b)$$

s = same as (2.2.2.2.61)

m_ϕ = same as (2.2.2.2.62)

$\gamma\dot{w}$ = same as (2.2.2.2.63)

$$\gamma\dot{v} = \frac{2Z}{R}[p_o(1 - e^{-\tau}) - p_s\tau]\,(-x^2 + 2x - u_2{}^2) \qquad (2.2.2.2.66)$$

c) $u_2 \leq x \leq 1$: regime 5-7

$$n_\phi = -1 \quad , \quad m_\theta = 1$$

$$n_\theta = -1 + \frac{2Z}{R}\left[\, \frac{8}{3}(p-p_s)x^3 - \frac{5}{2}(p-p_s - 1.2h)\; x^2 \,\right] \qquad (2.2.2.2.67)$$

s = same as (2.2.2.2.61)

m_ϕ = same as (2.2.2.2.62)

$\gamma\dot{w}$ = same as (2.2.2.2.63)

$$\gamma\dot{v} = \frac{4Z}{R}[\, p_o(1 - e^{-\tau}) - p_s\tau]\; x\,(1 - x) \qquad (2.2.2.2.68)$$

With these results, it has been verified that all the conditions of continuity are satisfied.

We have also verified that the equations of equilibrium are all satisfied within the limits of the shallow shell approximation. (equations 2.1.2..-2.1.2.3.)

From (2.2.2.2. 59) which is

$$u_2{}^2 = \frac{3\dot{C} - \dot{Q}}{C} u_1 - \frac{5}{3} u_1{}^2$$

and with $\dot{C}$ and $\dot{Q}$ from (2.2.2.2.49) and (2.2.2.2.47), we have:

$$u_2{}^2 = \frac{5[p_o(1 - e^{-\tau}) - p_s\tau - 1.2h\tau]}{2[p_o(1 - e^{-\tau}) - p_s\tau]} u_1 - \frac{5}{3} u_1{}^2$$

(2. 2.2.2.69)

Let:

$$f(\tau) = \frac{p_o(1 - e^{-\tau}) - p_s\tau - 1.2h\tau}{p_o(1 - e^{-\tau}) - p_s\tau} \qquad (2.2.2.2.70)$$

We can write (2.2.2.2.69) in the form:

$$u_2{}^2 = u_1{}^2 \left[\frac{5}{2} \frac{f(\tau)}{u_1} - \frac{5}{3} \right]$$

Let:

$$g = \left[\frac{5}{2} \frac{f(\tau)}{u_1} - \frac{5}{3} \right]^{1/2} \qquad (2.2.2.2.71)$$

Then we derive:

$$u_1 = \frac{15f(\tau)}{10+6g^2} \qquad (2.2.2.2.72)$$

$$u_2 = \frac{15f(\tau)g}{10+6g^2} \qquad (2.2.2.2.73)$$

and the solution of the system (2.2.2.2.59) and (2.2.2.2.60) is equivalent to the determination of the function g.

From (2.2.2.2.73) we can have $\dot{u}_2$ in terms of f, $\dot{f}$, g, $\dot{g}$. Substituting into (2.2.2.2.60) the expressions of u_1, u_2 in (2.2.2.2.72), (2.2.2.2.73) and $\dot{u}_2$ obtained by differentiation of u_2, $f(\tau)$ and therefore $\dot{f}(\tau)$ being known, we arrive to the following equation to determine g:

$$[p_o(1 - e^{-\tau}) - p_s\tau - 1.2h\tau][p_o(1 - e^{-\tau}) - p_s\tau](3g^2 - 5)\dot{g} =$$

$$(3g^2 + 5) \times \Big\{ (\tfrac{1}{12})(p_o e^{-\tau} - p_s)[p_o(1 - e^{-\tau}) - p_s\tau - 1.2h\tau](11g^2 + 5)/g - (\tfrac{1}{18})(p_o e^{-\tau} - p_s - 1.2h) \times [p_o(1 - e^{-\tau}) - p_s\tau](g^2 + g + 1)(6g^2 + 10)/[g(g+1)] - 1.2hp_o[1 - (1+\tau)e^{-\tau}]g \Big\} \qquad (2.2.2.2.74)$$

To determine the initial value g_o of the auxiliary variable g, we assume that the applied pressure p is maintained to the constant value p_o from 0 to T_o, and is allowed to decrease exponentially for $\tau > T_o$. Then for $0 \leq \tau \leq T_o$, g will have a constant value g_o and therefore $\dot{g} = 0$ and $\dot{f} = 0$ for $0 \leq \tau \leq T_o$. From (2.2.2.2.74), with $p=p_o e^{-\tau}$ and $p_o(1-e^{-\tau})$ replaced by p_o and $p_o\tau$ respectively, $\dot{g} = 0$, and noting that the last term $-1.2hp_o[1 - (1 + \tau)e^{-\tau}]$ comes from $\dot{f}$ which is zero in this case, we have the following equations to determine g_o:

$$\frac{1}{12}(p_o-p_s)(p_o-p_s-1.2h)\frac{11g_o{}^2+5}{g_o}$$

$$-\frac{1}{18}(p_o-p_s-1.2h)(p_o-p_s)\frac{(g_o{}^2+g_o+1)(6g_o^2+10)}{g_o(g_o+1)} = 0$$

which can be reduced to:

$$2(g_o{}^2+g_o+1)(6g_o{}^2+10) - 3(g_o+1)(11g_o{}^2+5) = 0$$

or, rearranging:

$$(g_o - 1)(12g_o{}^3 - 9g_o{}^2 - 10g_o - 5) = 0 \qquad (2.2.2.2.74a)$$

In the limit, when $T_o \rightarrow 0$, this argument is still valid, and the initial value of g is to be determined from (2.2.2.2.74a). The solution $g_o = 1$ corresponds to $u_1 = u_2$ and has been found to be unfit. Therefore the initial value g_o is the solution of:

$$12\, g_o{}^3 - 9g_o{}^2 - 10g_o - 5 = 0 \qquad (2.2.2.2.75)$$

which has one real root:

$$g_o \simeq 1.494 \qquad (2.2.2.2.76)$$

With the initial value g_o of g determined by equation (2.2.2.2.75), the initial value $\dot{g}_o$ of $\dot{g}$ at $\tau = 0^+$ from (2.2.2.2.74) will have the form $\frac{0}{0}$. Using the L'Hospital's rule to find the true limit of $\dot{g}_o$, we obtain:

$$\dot{g}_o = \frac{2.4p_o}{(p_o-p_s)(p_o-p_s-1.2h)} \times \frac{(6g_o^2+10)(3g_o^4-6g_o^3-g_o^2+5g_o+5)}{432g_o^5-414g_o^4+672g_o^3-900g_o^2-320g_o+50}$$

(2.2.2.2.76a)

With the initial values of g and $\dot{g}$ thus determined, we can then integrate equation (2.2.2.2.74) numerically. With g known for each value of τ, we can determine the corresponding values of u_1 and u_2 by using equations (2.2.2.2.72) and (2.2.2.2.73). (Sample output in Appendix A3)

We observe from the expression of $f(\tau)$ in (2.2.2.2.70) that:

$$f(0) = \lim_{\tau=0} \frac{p_o(1-e^{-\tau}) - p_s\tau - 1.2h\tau}{p_o(1-e^{-\tau}) - p_s\tau} = \frac{p_o-p_s - 1.2h}{p_o - p_s}$$

from which we have:

$$u_{10} = \frac{p_o-p_s - 1.2h}{p_o-p_s} \times \frac{15}{10 + 6g_o{}^2} \qquad (2.2.2.2.77)$$

$$u_{20} = \frac{p_o-p_s - 1.2h}{p_o-p_s} \times \frac{15g_o}{10 + 6g_o{}^2} \qquad (2.2.2.2.78)$$

From these expressions of u_{10} and u_{20}, we can see that u_{10} and u_{20} are positive only for $p_o-p_s > 1.2h$ and thus find again, by a different way, that for $p_o-p_s \leq 1.2h$, the whole shell collapses in one regime.

We can also see that the value τ_1 of τ for which:

$$u_1 = u_2 = 0$$

is determined by:

$$f(\tau_1) = 0 \qquad \text{or}$$

$$p_o(1-e^{-\tau_1}) - p_s\tau_1 - 1.2h\tau_1 = 0 \qquad (2.2.2.2.79)$$

At $\tau = \tau_1$, the collapse regime 5-7 of the interval $u_2 \leq x \leq 1$ spreads to the whole shell and the first stage of motion ends.

B. Second Stage of Motion, $\tau_1 \leq \tau \leq \tau_f$

For $\tau \geq \tau_1$, the whole shell collapse under regime 5-7 as in the low pressure case. The expressions for the stresses: n_θ, s, m_ϕ and for the acceleration of the central point $\gamma\ddot{w}_o$ remain the same, because we have the same equations of equilibrium (2.1.2.1)-(2.1.2.3) with the same known stresses: $n_\phi = -1$, $m_\theta = 1$, and the same boundary conditions.

Thus we have, from (2.2.2.1.11)-(2.2.2.1.13):

$$s = -\frac{6h - (p-p_s)}{2}x - \frac{2(p-p_s)}{3}x^2 \qquad (2.2.2.1.11)$$

$$n_\theta = -1 + \frac{2Z}{R}\left[\frac{8}{3}(p-p_s)x^3 - \frac{5}{2}(p-p_s-1.2h)x^2\right] \qquad (2.2.2.1.12)$$

$$m_\phi = 1 - \frac{6h - (p-p_s)}{6h}x^2 - \frac{p-p_s}{6h}x^3 \qquad (2.2.2.1.13)$$

and from (2.2.2.1.7):

$$\gamma\ddot{w}_o = 2(p-p_s) = 2(p_oe^{-\tau} - p_s)$$

The flow rules remain the same, and from (2.2.2.1.9), (2.2.2.1.10), we have:

$$\gamma\dot{w} = \gamma\dot{w}_o(1-x)$$

$$\gamma\dot{v} = \frac{2Z}{R}\gamma\dot{w}_o x(1-x)$$

To determine $\gamma\dot{w}_o$, we integrate $\gamma\ddot{w}_o$ from τ_1 to τ:

$$\gamma\dot{w}_o(\tau) - \gamma\dot{w}_o(\tau_1) = \int_{\tau_1}^{\tau} \gamma\ddot{w}_o d\tau = 2\int_{\tau_1}^{\tau} (p_o e^{-\tau} - p_s) d\tau$$

$$\gamma\dot{w}_o(\tau) - \gamma\dot{w}_o(\tau_1) = 2[p_o(e^{-\tau_1} - e^{-\tau}) - p_s(\tau-\tau_1)]$$

From (2.2.2.2.63), we have:

$$\gamma\dot{w}_o(\tau_1) = \gamma\dot{w}_o(\tau_1^-) = 2[p_o(1-e^{-\tau_1}) - p_s\tau_1$$

Therefore:

$$\gamma\dot{w}_o(\tau) = 2[p_o(e^{-\tau_1} - e^{-\tau}) - p_s(\tau-\tau_1)] + 2[p_o(1-e^{-\tau}) - p_s\tau_1]$$

$$\gamma\dot{w}_o(\tau) = 2[p_o(1-e^{-\tau}) - p_s\tau]$$

With this, we have:

$$\gamma\dot{w} = 2[p_o(1-e^{-\tau}) - p_s\tau](1-x) \qquad (2.2.2.1.9)$$

$$\gamma\dot{v} = \frac{4Z}{R}[p_o(1-e^{-\tau}) - p_s\tau]\ x(1-x) \qquad (2.2.2.1.10)$$

These expressions are exactly those obtained in the case of low pressure (2.2.2.1.9) and (2.2.2.1.10).

We can conclude that for $\tau \geq \tau_1$, the expressions for the stresses and velocities are the same as in the low pressure case.

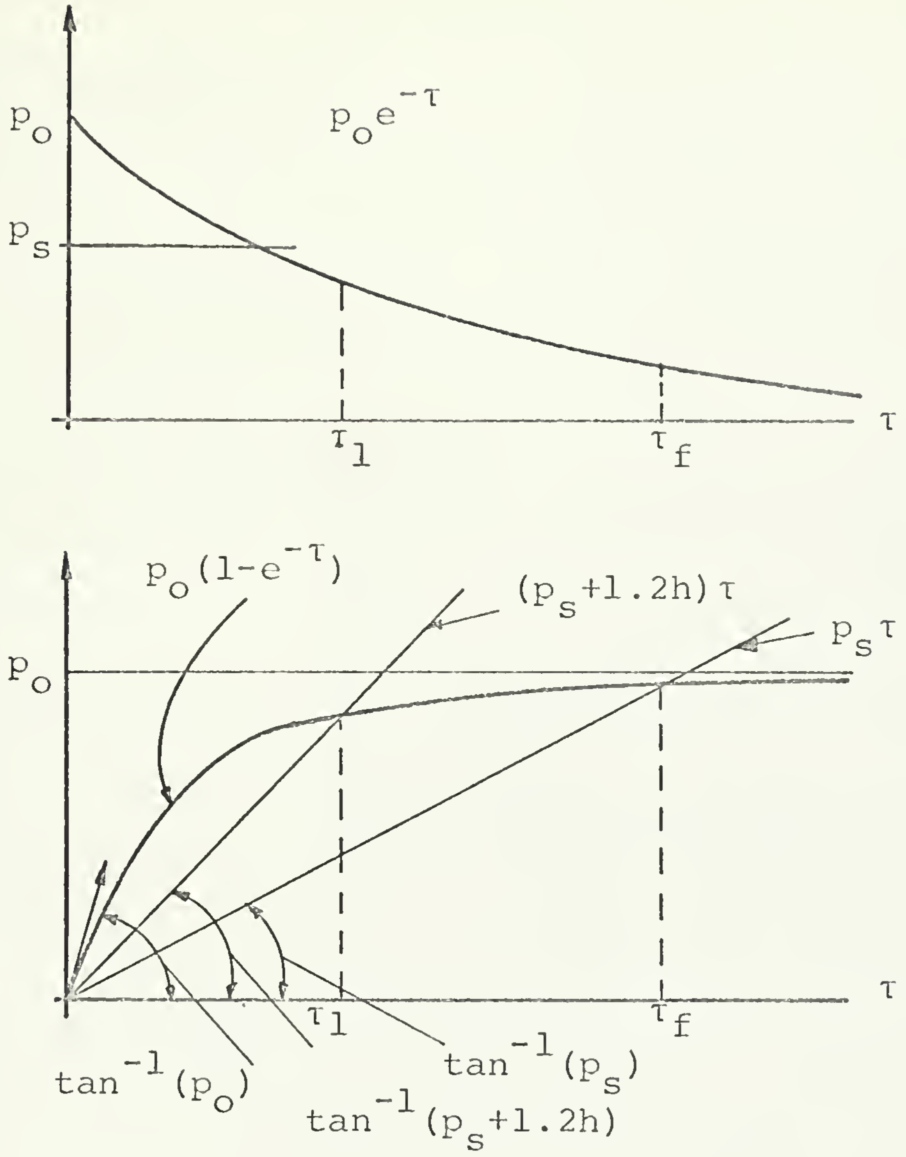
By the way, we note that these expressions are also those corresponding to the interval $u_2 \leq x \leq 1$ in the first stage of motion. Thus after spreading to the whole shell at $\tau = \tau_1$, the solution for this regime remains valid afterwards, which is what we expect, and which ensures the continuity of the solution with respect to τ.

At $\tau = \tau_f$ determined by

$$p_o(1-e^{-\tau_f}) - p_s\tau_f = 0 \qquad (2.2.2.2.80)$$

which is the same as (2.2.2.1.14) we have $\gamma\dot{w} = \gamma\dot{v} = 0$ and the motion ceases.

We can visualize the relative portions of τ_1 determined by (2.2.2.2.79) and τ_f determined by (2.2.2.2.80) above in the following graph:



Dynamic Pressure Profile (a)

Relative Positions of τ_1 & τ_f (b)

Fig. 11

C. Admissibility of the Solution

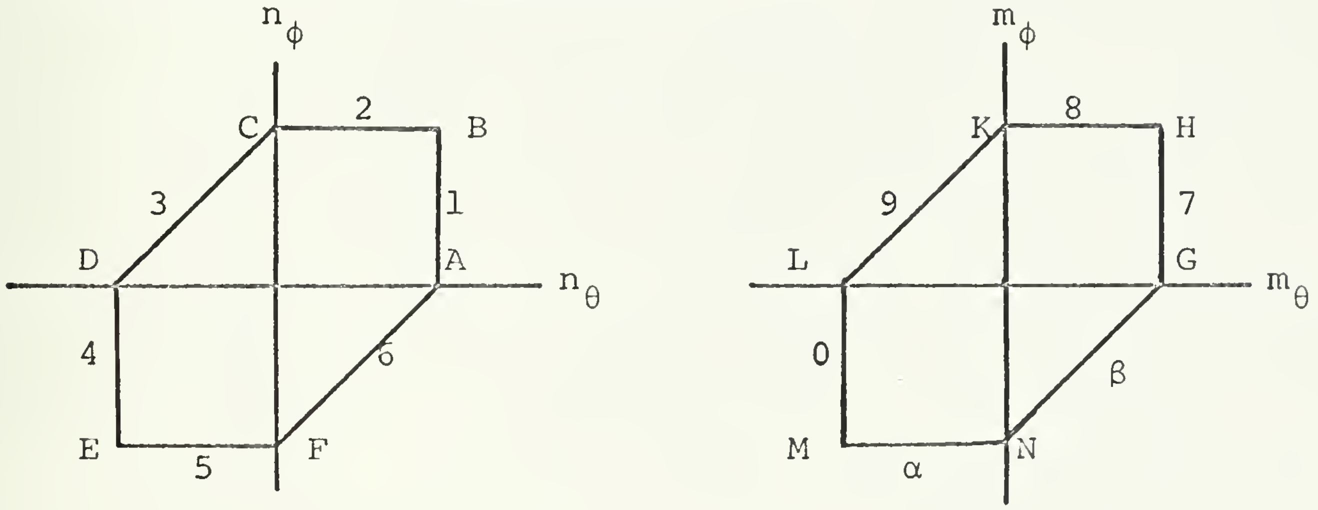


Fig. 12

1. Kinematic Admissibility

a) $0 \leq \tau \leq \tau_1$

i) $0 \leq x \leq u_1$: regime E-7

Conditions for kinematic admissibility:

$$\dot{e}_\theta \leq 0 \qquad (2.2.2.2.81)$$

$$\dot{e}_\phi \leq 0 \qquad (2.2.2.2.82)$$

$$\dot{k}_\theta \geq 0 \qquad (2.2.2.2.83)$$

From (2.1.1.1) and with $\dot{w}$ and $\dot{v}$ from (2.2.2.2.63) and (2.2.2.2.64) respectively, we have:

$$\gamma\dot{e}_\theta = \frac{2Z}{R}\left\{\frac{8}{3}[p_o(1-e^{-\tau}) - p_s\tau]x - \frac{5[p_o(1-e^{-\tau})-p_s\tau - 1.2h\tau]}{2}\right\}$$

(2.2.2.2.84)

since for $\tau \leq \tau_1$, $p_o(1-e^{-\tau}) - p_s\tau - 1.2h\tau \geq 0$ and

$$p_o(1-e^{-\tau}) - p_s\tau > 0,$$

$\dot{e}_\theta$ will be negative if

$$x \leq \frac{15}{16}\,\frac{p_o(1-e^{-\tau}) - p_s\tau - 1.2h\tau}{p_o(1-e^{-\tau}) - p_s\tau}$$

(2.2.2.2.85)

From (2.1.1.3), we have, similarly:

$$\gamma\dot{e}_\phi = \frac{2Z}{R}\left\{\frac{10}{3}[p_o(1-e^{-\tau}) - p_s\tau]x - \frac{5}{2}[p_o(1-e^{-\tau}) - p_s\tau - 1.2h\tau]\right\}$$

(2.2.2.2.86)

$\dot{e}_\phi$ will be negative if

$$x \leq \frac{15}{20}\,\frac{p_o(1-e^{-\tau}) - p_s\tau - 1.2h\tau}{p_o(1-e^{-\tau}) - p_s\tau} \qquad (2.2.2.2.87)$$

From (2.1.1.2), with $\dot{w}$ and $\dot{v}$ from (2.2.2.2.63) and (2.2.2.2.64) respectively, and using the shallow shell approximation, we have:

$$\gamma\dot{K}_\theta = \frac{2h}{x}[p_o(1-e^{-\tau}) - p_s\tau] \qquad (2.2.2.2.88)$$

From this expression, we can see that $\gamma\dot{K}_\theta$ is positive for $0 \leq \tau \leq \tau_f$ and thus (2.2.2.2.83) is satisfied. Comparing (2.2.2.2.85) and (2.2.2.2.87), and noticing that $0 \leq x \leq u_1$, we see that both conditions (2.2.2.2.81) and (2.2.2.2.82) are simultaneously satisfied if we have:

$$u_1 \leq \frac{1.5}{20}\,\frac{p_o(1-e^{-\tau}) - p_s\tau - 1.2h\tau}{p_o(1-e^{-\tau}) - p_s\tau} \qquad (2.2.2.2.89)$$

A numerical approach has been used for p_o-p_s having the values 2.0, 4.0, 6.0 and for the ratio of (Z/R)/h having the values 4.0, 8.0, 12.0, 16.0, 20.0 (which is equivalent to the ratio of depth/thickness having the values 1.0, 2.0, 3.0, 4.0, 5.0). It has been found that condition (2.2.2.2.-89) is always satisfied; thus, for this range of τ and x, the solution is kinematically admissible. (Appendix A3)

ii) $u_1 \leq x \leq u_2$: regime 4-7

Conditions for kinematic admissibility:

$$\dot{e}_\theta \leq 0 \qquad (2.2.2.2.90)$$

$$K_\theta \geq 0 \qquad (2.2.2.2.91)$$

With $\dot{w}$ and $\dot{v}$ from (2.2.2.2.63) and (2.2.2.2.66) respectively, we have:

$$\gamma\dot{e}_{\theta} = -\frac{2Z}{R}[p_o(1-e^{-\tau}) - p_s\tau]\ \frac{u_2{}^2-x^2}{x}$$

Since $x \leq u_2$, we have $\dot{e}_{\theta} \leq 0$.

With $\dot{w}$ and $\dot{v}$ from (2.2.2.2.63) and (2.2.2.2.66) respectively, and using the shallow shell approximation, we find again:

$$\gamma\dot{K}_{\theta} = \frac{2h}{x}[p_o(1-e^{-\tau}) - p_s\tau]$$

which is positive.

Thus, in the interval (u_1,u_2), the solution is admissible provided u_1 and u_2 exist and such that $0 \leq u_1 \leq u_2 \leq 1$. A numerical approach has been used to solve for u_1 and u_2 and it has been found that they do exist and are in that order. (Sample output in Appendix A3)

iii) $u_2 \leq x \leq 1$: regime 5-7

The solution for this interval is the same as in the low pressure case. It has been proved that the solution is kinematically admissible for $\tau < \tau_f$.

Therefore the solution is kinematically admissible for $0 \leq \tau \leq \tau_1$.

b) $\tau_1 \leq \tau \leq \tau_f$: regime 5-7

The solution for this stage of motion is the same as that for the low pressure case. This solution has been proved to be kinematically admissible; therefore the

solution is kinematically admissible for $0 \leq \tau \leq \tau_f$.

2. Static Admissibility

a) n_ϕ:

Condition for admissibility: $-1 \leq n_\phi \leq 0$

For $0 \leq x \leq u_1$ and $u_2 \leq x \leq 1$, $n_\phi = -1$

For $u_1 \leq x \leq u_2$, we have

from (2.2.2.2.65b):

$$n_\phi = -1 + \frac{5}{6}\,\frac{Z}{R}\,\frac{(x-u_1)(u_2-x)}{x}\,P(x) \qquad (2.2.2.2.92)$$

where:

$$P(x) = (p-p_s)x^2 - [2(p-p_s - 1.2h) - (p-p_s)(u_1+u_2)]x$$
$$- [2(p-p_s - 1.2h) - (p-p_s)(u_1+u_2)]\,\frac{u_1u_2}{u_1+u_2}$$

P(x) is a polynomial of 2nd degree of the form

$$P(x) = Ax^2 - Cx - D$$

From this expression of n_ϕ, we see that a necessary condition for n_ϕ to be admissible is that P(x) be positive.

If S is the sum of the roots of

$$P(x) = 0\ , \quad \text{we have}$$

$$S = \frac{C}{A}$$

The numerical approach which computes u_1 and u_2 also evaluates $P(u_1)$, $P(u_2)$, and

$$(DD1) \equiv u_1 - \frac{S}{2}$$

$$(DD2) \equiv u_2 - \frac{S}{2}$$

It has been found that for the range of pressure: $1.2h \leq p_o - p_s \leq 6h$, and for the height of shell ranging from 1 to 5 times its thickness, (DD1) and (DD2) have same signs and $P(u_2) > P(u_1) > 0$.

Since (DD1) and (DD2) are both positive or both negative, u_1 and u_2 are on the same side of S/2. We have thus the following disposition, according to the sign of $A = p - p_s$:

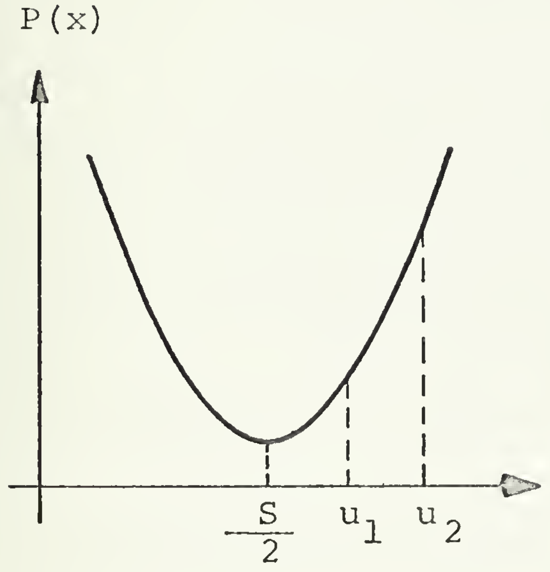


Fig. 13a

$A = p - p_s > 0$ and $P(x) = 0$ has imaginary roots

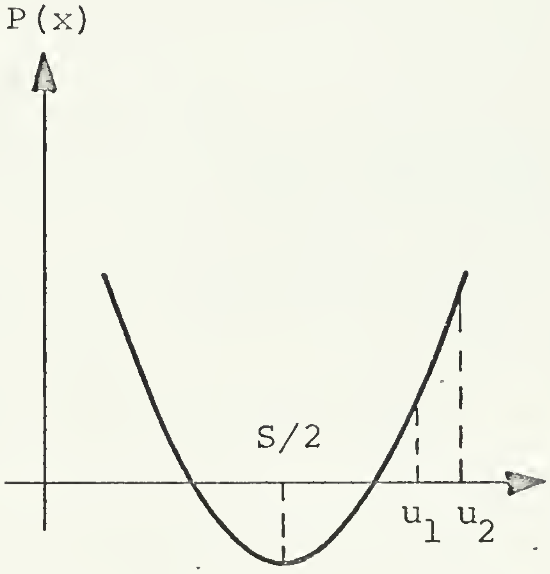


Fig. 13b

$A = p - p_s > 0$ and $P(x) = 0$ has real roots

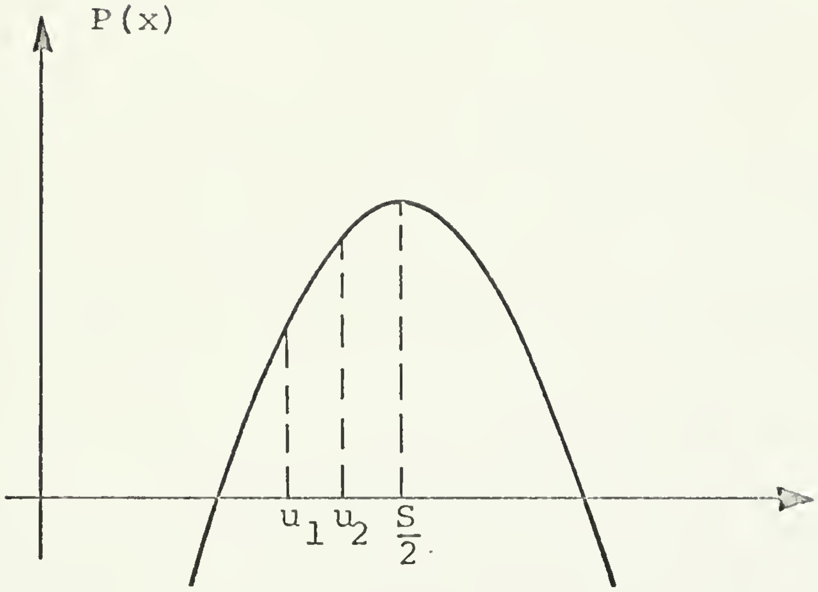


Fig. 13c

$A = p - p_s < 0$

thus in any case, $P(x) > 0$ for $u_1 \le x \le u_2$.

The maximum value $P(u_2)$ of $P(x)$ is of the order 2h.

From the expression of n_ϕ in (2.2.2.2.92) we have:

$$(n_\phi)_{max} \le -1 + \frac{5}{6}\frac{Z}{R}\left[\frac{(x-u_1)(u_2-x)}{x}\right]_{max} x\,[P(x)]_{max}$$

It can be proved that:

$$\left[\frac{(x-u_1)(u_2-x)}{x}\right]_{max} = (\sqrt{u_2} - \sqrt{u_1})^2$$

Since $0 < u_1 < u_2 < 1$, $(\sqrt{u_2} - \sqrt{u_1})^2 \le 1$

and $[P(x)]_{max} = P(u_2) \simeq 2h$,

we have:

$$(n_\phi)_{max} \leq -1 + \frac{5}{3}\frac{Z}{R} h$$

Since Z/R and h are smaller than 1, we see that $(n_\phi)_{max} < 0$. Thus n_ϕ is admissible for regime 4-7.

b) m_ϕ:

Condition for admissibility:

$$0 \leq m_\phi \leq 1$$

m_ϕ has the same expression as in the case of low pressure range, and it has been proved that m_ϕ is admissible for:

$$p_o - p_s \leq 6h \qquad \text{(condition (2.2.2.1.17))}$$

and $$12h + p_o e^{-\tau f} - p_s \geq 0 \qquad \text{(condition (2.2.2.1.19))}$$

Condition (2.2.2.1.19) has been found numerically to be satisfied for the range of pressure: $p_o - p_s \leq 6h$ and for the height of the shell varying from 1 to 5 times its thickness. (Sample output in Appendix A3)

Therefore m_ϕ is admissible for $p_o - p_s \leq 6h$.

c) n_θ:

Condition for admissibility:

$$-1 \leq n_\theta \leq 0$$

From the expression of n_θ in (2.2.2.2.67), we can write n_θ in the form:

$$n_\theta = -1 + \frac{2Z}{R}x^2 \times S(x) \qquad (2.2.2.2.93)$$

where

$$S(x) = \frac{8}{3}(p-p_s)x - \frac{5}{2}(p-p_s - 1.2h) \qquad (2.2.2.2.94)$$

For $n_\theta \geq -1$, we must have

$$S(x) \geq 0$$

i) If $p-p_s > 0$, $S(x)$ is monotonically increasing function of x, and $S(x)$ is positive for $u_2 \leq x \leq 1$ if $S(u_2)$ is:

$$(AD2) \equiv S(u_2) = \frac{8}{3}(p_o e^{-\tau} - p_s)u_2 - \frac{5}{2}(p-p_s - 1.2h) \geq 0 \qquad (2.2.2.2.95)$$

The maximum of n_θ occurs at $x = 1$, and its value is:

$$(n_\theta)_{max} = -1 + \frac{2Z}{R}[\frac{8}{3}(p-p_s) - \frac{5}{2}(p-p_s - 1.2h)] = -1 + \frac{2Z}{R}[\frac{p-p_s}{6} + 3h]$$

Since $p_o-p_s \leq 6h$, we have

$$(n_\theta)_{max} \leq -1 + 2\frac{Z}{R}(h+3h) = -1 + 8h\frac{Z}{R}$$

Since h and $\frac{Z}{R}$ are small compared to 1, we see that, we have:

$$(n_\theta)_{max} < 0$$

ii) If $p-p_s < 0$, $S(x)$ is monotonically decreasing function of x and $S(x)$ is positive for $u_2 \leq x \leq 1$ if $S(1)$ is:

$$S(1) = \frac{8}{3}(p-p_s) - \frac{5}{2}(p-p_s - 1.2h) \geq 0$$

or $$\frac{p-p_s}{6} + 3h \geq 0$$

or $$p-p_s + 18h \geq 0$$

Since p decreases with τ, this condition is satisfied for $\tau < \tau_1$ if it is at $\tau = \tau_1$:

$$p_o e^{-\tau_1} - p_s + 18h \geq 0$$

If we compare this condition with condition (2.2.2.1.19) and notice that $\tau_f > \tau_1$ and hence $p_o e^{-\tau_1} > p_o e^{-\tau_f}$, we can see that it is always satisfied if (2.2.2.1.19) is.

From the expression of n_θ (2.2.2.2.67), we have:

$$n'_\theta = \frac{2Z}{R}[8(p-p_s)x - 5(p-p_s - 1.2h)]x$$

and $n'_\theta = 0$

for $$x_n = \frac{5(1.2h+p_s-p)}{8(p_s-p)}$$

and n_θ has an extremum whose value is:

$$(n_\theta)_{ext} = -1 + \frac{2Z}{R}x_n{}^2 \left[\frac{5(1.2h+p_s-p)}{6} \right]$$

This extremum is larger than -1. For this extremum to be admissible, we must have also:

$$(n_\theta)_{ext} < 0 \qquad \text{or}$$

$$\frac{2Z}{R}x_n^2 \times \frac{5(1.2h+p_s-p)}{6} < 1 \text{ or,}$$

since $x_n^2 \leq 1$, this condition will be satisfied if we have:

$$\frac{2Z}{R} \times \frac{5(1.2h+p_s-p)}{6} < 1 \qquad (2.2.2.2.96)$$

From condition (2.2.2.1.19), we deduce:

$$p_s-p \leq p_s-p_o e^{-\tau_f} \leq 12h$$

With this result, we have:

$$\frac{2Z}{R} \times \frac{5(1.2h+p_s-p)}{6} \leq 22h\frac{Z}{R}$$

Therefore condition (2.2.2.2.96) will be satisfied if:

$$22h \times \frac{Z}{R} < 1$$

This condition is precisely condition (2.2.2.1.25) and we know that it is satisfied for a thin, shallow shell.

The numerical method which gives u_1 and u_2 has also found that condition (AD2) defined in (2.2.2.2.95) is satisfied. (Sample output in Appendix A3)

For $\tau_1 \leq \tau \leq \tau_f$, n_θ has the same expression, but now it has an extremum at $x = 0$. To be admissible, this

extremum must be minimum or:

$$(n''_\theta)_{x=0} \geq 0 \quad \text{or:} \quad p-p_s \leq 1.2h$$

Since p decreases monotonically with τ, this condition will be satisfied for $\tau > \tau_1$ if it is at $\tau = \tau_1$:

$$p-p_s \leq p_o e^{-\tau_1} - p_s \leq 1.2h \qquad (2.2.2.2.97)$$

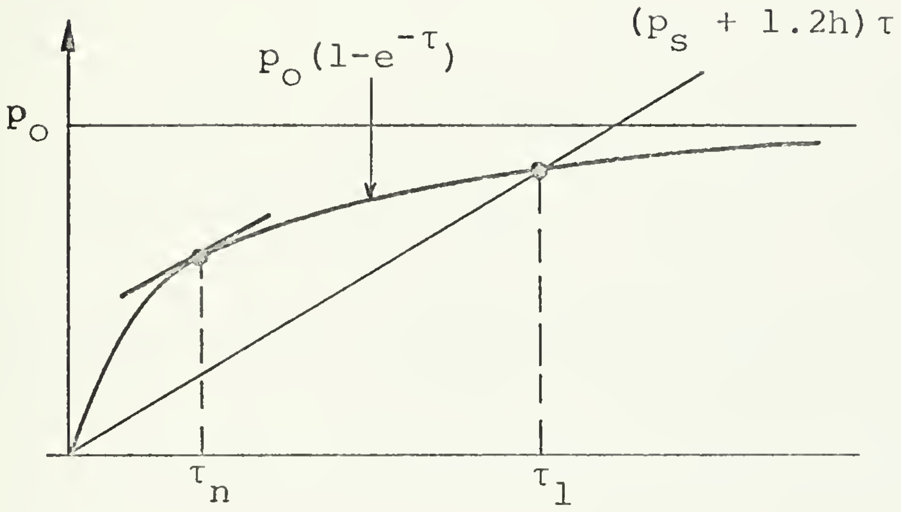


Fig. 14

Relative Positions of τ_n and τ_1

The function $p_o(1-e^{-\tau})$ has a derivative equal to $p_o e^{-\tau}$

From the law of the mean in mathematical analysis, there is a value τ_n of τ: $0 < \tau_n < \tau_1$, such that:

$$\frac{p_o(1-e^{-\tau_1})}{\tau_1} = p_o e^{-\tau_n}$$

From (2.2.2.2.79), we have:

$$p_o(1-e^{-\tau_1}) = (p_s + 1.2h)\tau_1$$

Therefore:

$$p_o e^{-\tau_n} = p_s + 1.2h, \quad (0 < \tau_n < \tau_1)$$

Since $\tau_n < \tau_1$ and since $p_o e^{-\tau}$ decreases monotonically, we have:

$$p_o e^{-\tau} < p_o e^{-\tau_n} = p_s + 1.2h \quad \text{or:}$$

$$p_o e^{-\tau_1} - p_s < 1.2h$$

and condition (2.2.2.2.97) is always satisfied.

Thus, we have found that n_θ is admissible for $0 \leq \tau \leq \tau_f$.

3. Conclusion

The solution is found to be kinematically and statically admissible and is therefore an exact one according to the yield surface which has been selected.

D. Maximum Central Deflection and Energy Absorbed.

Since $\gamma\dot{w}$ has the same expression as in the case of low pressure in both stages of motion ($0 \leq \tau \leq \tau_1$ and $\tau_1 \leq \tau \leq \tau_f$) the central deflection and the energy absorbed

will have also the same expressions: (the difference is that now $p_o - p_s$ can go up to 6h)

From (2.2.2.1.27a):

$$\gamma w_o = 2[(p_o - p_s)\tau_f - \frac{1}{2} p_s \tau_f^2] \qquad (2.2.2.1.27$$

From (2.2.2.1.28):

$$E_{abs} = \frac{2\pi R^2 N_o}{3\gamma} \left\{ \frac{p_o^2}{2}(1 - e^{-\tau_f})^2 - p_o p_s [1 - (1+\tau_f) e^{-\tau_f}] \right\} \qquad (2.2.2.1.28)$$

E. Final Displacement Distribution.

1. Normal Displacement Component.

By integrating (2.2.2.2.63) with respect to τ, and observing that at $\tau = 0$, $\gamma w = 0$, the final normal displacement component γw_f can be written as:

$$\gamma w_f = 2[(p_o - p_s)\tau_f - \frac{1}{2} p_s \tau_f^2](1-x) \qquad (2.2.2.2.98)$$

2. Tangential Displacement.

The expression for the tangential displacement γv_f depends on the position of x.

If u_{10} and u_{20} are the initial values of u_1 and u_2 respectively at $\tau = 0$, we have:

a) $u_{20} \leq x \leq 1$: integrating (2.2.2.2.64) with respect to τ, we have:

$$\gamma v_f = \frac{4Z}{R}[p_o - p_s)\tau_f - \frac{1}{2}p_s\tau_f^{\,2}]x(1-x) \qquad (2.2.2.2.99a)$$

b) $u_{10} \leq x \leq u_{20}$: using the velocity from (2.2.2.2.64) and (2.2.2.2.66), we have:

$$\gamma v_f = \frac{2Z}{R}\int_0^{\tau_{x2}} [p_o(1-e^{-\tau}) - p_s\tau](2x - x^2 - u_2^{\,2})\,d\tau$$

$$+ \frac{4Z}{R}\int_{\tau_{x2}}^{\tau_f} [p_o(1-e^{-\tau}) - p_s\tau]x(1-x)\,d\tau \qquad (2.2.2.2.99b)$$

where τ_{x2} is the value of τ for which $u_2 = 2$:

$$u_2(\tau_{x2}) = x$$

τ_{x2} can be found numerically.

c) $0 \leq x \leq u_{10}$: using the velocity from (2.2.2.2.64), (2.2.2.2.66), and (2.2.2.2.68):

we have:

$$\gamma v_f = \frac{2Z}{R}x\int_0^{\tau_{x1}}\left\{\frac{6h\tau - [p_o(1-e^{-\tau}) - p_s\tau]}{2} + \frac{2[p_o(1-e^{-\tau}) - p_s\tau]}{3}\,x\right\}d\tau$$

$$+ \frac{2Z}{R}\int_{\tau_{x1}}^{\tau_{x2}} [p_o(1-e^{-\tau}) - p_s\tau](2x - x^2 - u_2^{\,2})\,d\tau$$

$$+ \frac{4Z}{R}\int_{\tau_{x2}}^{\tau_{x1}} [p_o(1-e^{-\tau}) - p_s\tau]x(1-x)\,d\tau$$

(2.2.2.2.99c)

τ_{x1} and τ_{x2} are the values of τ for which we have respectively: $x = u_1$ and $x = u_2$:

$$u_1(\tau_{x1}) = x$$

$$u_2(\tau_{x2}) = x$$

These expressions of γv_f are rather complicated. Since γv_f is not important compared to γw_f, approximately

$$\frac{\gamma v_f}{\gamma w_f} = 0(\frac{Z}{R})$$

and Z/R is small for shallow shell, we do not evaluate γv_f numerically.

2.2.2.3 Medium High Pressure Range $6h \leq p_o - p_s \leq \lambda h$

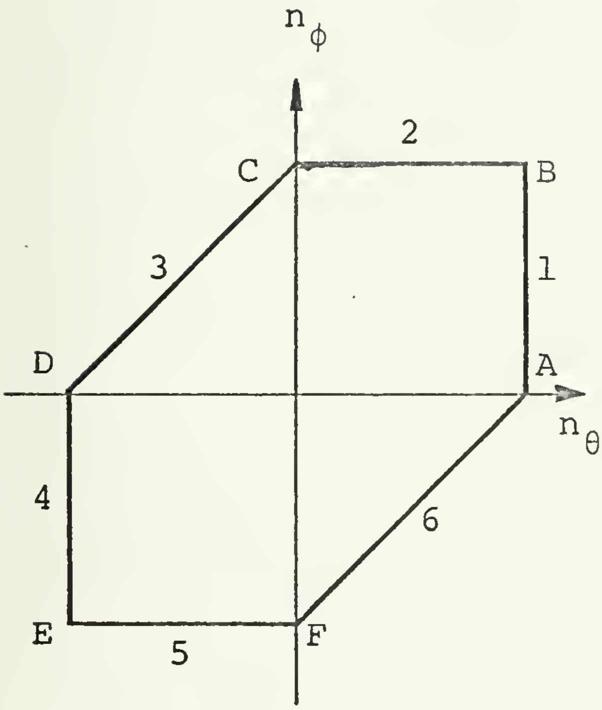
A. First Stage of Motion, $0 \leq \tau \leq \tau_o$

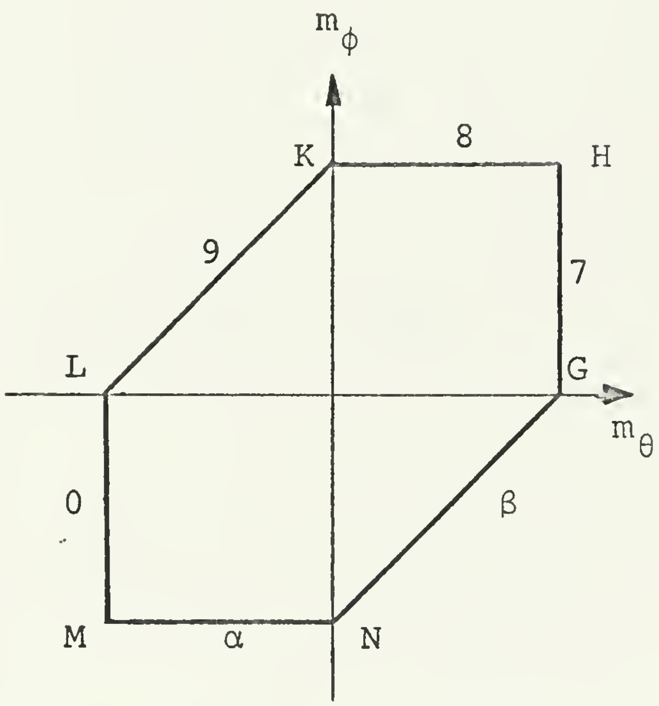
When $p_o - p_s > 6h$, then from the expression of m_ϕ in (2.2.2.2.62):

$$(m'_\phi)_{x=0} = 0$$

$$(m''_\phi)_{x=0} = \frac{p - p_s - 6h}{3h} > 0$$

which means that there is a minimum at $x = 0$. Since at $x = 0$, $m_\phi = 1$, this also means that m_ϕ will increase from 1, which is not possible for rigid perfectly plastic material.





We therefore assume these collapse regimes for the shell:

$0 \leq x \leq u_o$: regime E-H
$u_o \leq x \leq u_1$: regime E-7
$u_1 \leq x \leq u_2$: regime 4-7
$u_2 \leq x \leq 1$: regime 5-7

The solutions are then:

1) $0 \leq x \leq u_o$: regime E - H

$$n_\phi = n_\theta = -1$$

$$m_\theta = m_\phi = 1$$

Then from (2.1.2.1):

$$xs = 0 \quad , \quad s = 0$$

From (2.1.2.2):

$$\gamma\ddot{v} = 0, \qquad (2.2.2.3.1) \quad , \quad \text{then:}$$

$$\gamma\dot{v} = 0, \qquad (2.2.2.3.2)$$

and from (2.1.2.3):

$x\gamma\ddot{w} = (p - \frac{4Z}{R})x$, or with $\ddot{Q}$ defined in (2.2.2.2.35)

$\gamma\ddot{w} = \ddot{Q}$ (2.2.2.3.3), then

$$\gamma\dot{w} = \dot{Q} \qquad (2.2.2.3.4)$$

where $\dot{Q}$ has been defined in (2.2.2.2.47).

2) $u_o \leq x \leq u_1$: regime E-7

The equations of equilibrium and the flow rules are the same as those corresponding to the interval $0 \leq x \leq u_1$, first stage of motion, of the medium pressure case.

Therefore the resulting equation of the system in terms of $Y = xs$ is the same as in (2.2.2.2.9) which is:

$$\xi^2 \frac{d^2Y}{d\xi^2} - \xi \frac{dY}{d\xi} - \xi^2 Y = \frac{\ddot{C}_o}{8Z^3/R^3} \xi^3$$

But now, we have only one boundary condition: at $x = u_o$, $Y = 0$, since $s = 0$ at $x = u_o^-$ and s must be continuous.

The exact solution has been found (page 66) to be, with $\ddot{C}_2$ now different from zero:

$$Y = \frac{\ddot{C}_o}{8Z^3/R^3}\left[-\xi + \xi I_1 \int_o^\xi K_o d\eta + \xi K_1 \int_o^\xi I_o d\eta\right] + \ddot{C}_1(\xi I_1) + \ddot{C}_2(\xi K_1)$$

$$\xi = \frac{2Z}{R} x \qquad (2.2.2.3.5)$$

Expanding and using the shallow shell approximation, we obtain:

$$Y = \frac{2Z^2}{R^2} \ddot{C}_1 x^2 + \frac{1}{3} \ddot{C}_o x^3 + \ddot{C}_2$$

Using the boundary condition:

$$Y(u_o) = 0,$$

we obtain:

$$\ddot{C}_2 = - \frac{2Z^2}{R^2} \ddot{C}_1 u_o^2 - \frac{1}{3} \ddot{C}_o u_o^3 \qquad (2.2.2.3.6)$$

With this, we have then:

$$Y = xs = \frac{2Z^2}{R^2}\ddot{C}_1(x^2-u_o{}^2) + \frac{1}{3}\ddot{C}_o(x^3-u_o{}^3) \qquad (2.2.2.3.7)$$

From equation (2.2.2.2.5) we have, with Y from (2.2.2.3.5):

$$x\gamma\ddot{w} = \ddot{Q}x + \frac{\ddot{C}_o}{4Z^2/R^2}\left[\xi I_o \int_o^\xi K_o d\eta - K_o \int_o^\xi I_o d\eta\right]$$

$$+ \frac{2Z}{R}\ddot{C}_1(\xi I_o) - \frac{2Z}{R}\ddot{C}_2(\xi K_o) \qquad (2.2.2.3.8)$$

Expanding and using shallow shell approximation, we obtain:

$$x\gamma\ddot{w} = \ddot{Q}x + \ddot{C}_o x^2 + \frac{4Z^2}{R^2}\ddot{C}_1 x + \frac{4Z^2}{R^2}\ddot{C}_2(\gamma_E + \log\frac{Zx}{R})x$$

where γ_E is the Euler's constant defined in page 70

With $\ddot{C}_2$ from (2.2.2.3.6), we have:

$$\gamma\ddot{w} = \ddot{Q} + \ddot{C}_o[1-\frac{4}{3}\frac{Z^2}{R^2}\frac{u_o{}^3}{x}(\gamma_E + \log\frac{Z}{R}x]x$$

$$+ \frac{4Z^2}{R^2}\ddot{C}_1 [1 - \frac{2Z^2}{R^2}u_o{}^2(\gamma_E + \log\frac{Z}{R}x)]$$

Since $\frac{Z}{R}\log\frac{Z}{R}$ is small when $\frac{Z}{R}$ is small, and $\gamma_E \simeq 0.5772$, we see that we can neglect $\frac{4}{3}\frac{Z^2}{R^2}\frac{u_o{}^3}{x}(\gamma_E+\log\frac{Zx}{R})$ and $\frac{2Z^2}{R^2}u_o^2(\gamma_E+\log\frac{Zx}{R})$ with respect to 1.

Therefore, we have:

$$\gamma \ddot{w} = \ddot{Q} + \frac{4Z^2}{R^2} \ddot{C}_1 + \ddot{C}_o x \qquad (2.2.2.3.9)$$

This expression can also be obtained by direct substitution of Y from (2.2.2.3.7) into (2.2.2.2.5), and which is what we will do for $\ddot{\gamma} v$ and m_ϕ since they can be obtained by algebraic substitution or integration of Y.

By integrating (2.2.2.3.9) with respect to τ from 0 to τ, we have:

$$\gamma \dot{w} = \dot{Q} + \frac{4Z^2}{R^2} \dot{C}_1 + \dot{C}_o x \qquad (2.2.2.3.10)$$

The initial condition $(\gamma \dot{w})_{\tau=0} = 0$ will be satisfied if we choose $\dot{C}_1(0) = \dot{C}_o(0) = 0$.

From the equilibrium equation (2.2.2.2.2), we have:

$$\gamma \ddot{v} = - \frac{2Z}{R} \left[\frac{2Z^2}{R^2} \ddot{C}_1 \frac{x^2 - u_o^{\,2}}{x} + \frac{1}{3} \ddot{C}_o \frac{x^3 - u_o^{\,3}}{x} \right] \qquad (2.2.2.3.11)$$

Integrating with respect to τ from 0 to τ, noticing that u_o is a function of τ, we obtain:

$$\gamma \dot{v} = - \frac{2Z}{R} \left[\frac{2Z^2}{R^2} \dot{C}_1 x + \frac{1}{3} \dot{C}_o x^2 - \frac{1}{x} \int_0^\tau \left(\frac{2Z^2}{R^2} \ddot{C}_1 + \frac{1}{3} \ddot{C}_o u_o \right) u_o^{\,2} d\tau \right] \qquad (2.2.2.3.12)$$

As before, the initial condition: $\tau = 0$, $\gamma \dot{v} = 0$ will be satisfied if we choose $\dot{C}_1(0) = \dot{C}_o(0) = 0$.

From the equilibrium equation (2.1.2.1), with $m_\theta = 1$ we obtain, after an integration from u_o to x, with $m\,(u_o) = 1$:

$$hxm_\phi = hx + \frac{2}{3}\frac{Z^2}{R^2}\ddot{C}_1(x^3 - 3u_o{}^2x + 2u_o{}^3)$$

$$+ \frac{1}{12}\ddot{C}_o(x^4 - 4u_o{}^3x + 3u_o{}^4) \qquad (2.2.2.3.13)$$

3) $u_1 \leq x \leq u_2$: regime 4-7

Here again, the equilibrium equations and the flow rules are exactly the same as those corresponding to the same interval, first stage of motion, of the medium pressure case.

Therefore equations (2.2.2.2.37)-(2.2.2.2.41) are still valid in this case and we have:

$$\gamma\dot{v} = \frac{\dot{A} + \frac{2Z}{R}\dot{B}}{2Z/R} + \frac{2Z}{R}\,x\,(\dot{C} - \frac{Z}{R}\dot{B}x) \qquad (2.2.2.2.37)$$

$$\gamma\dot{w} = \dot{C} - \frac{2Z}{R}\dot{B}x \qquad (2.2.2.2.38)$$

$$xn_\phi = \frac{\ddot{A} + \frac{8Z^3}{R^3}\ddot{D}}{8Z^3/R^3} + \frac{\frac{4Z^2}{R^2}\ddot{E} + \frac{2Z}{R} - p}{2Z/R}\,x + \frac{1}{2}\,\frac{\frac{2Z}{R}\ddot{B} - \frac{8Z^3}{R^3}\ddot{D}}{2Z/R}\,x^2$$

$$+ \frac{1}{3}\,\frac{Z}{R}\,(3\ddot{C} - \frac{4Z^2}{R^2}\ddot{E})x^3 + \frac{1}{12}\,\frac{Z}{R}\,(\frac{8Z^3}{R^3}\ddot{D} - 12\,\frac{Z}{R}\ddot{B})\;x^4$$

(2.2.2.2.39)

$$xs = \frac{\frac{4Z^2}{R}\ddot{E} - \ddot{Q}}{4Z^2/R^2} - \frac{\ddot{A} + \frac{8Z^3}{R^3}\ddot{D}}{4Z^2/R^2}\, x + \frac{1}{2}\left(\ddot{C} - \frac{4Z^2}{R^2}\ddot{E}\right) x^2$$
$$+ \frac{1}{6}\left(8\frac{Z^3}{R^3}\ddot{D} - 6\frac{Z}{R}\ddot{B}\right) x^3 \qquad (2.2.2.2.40)$$

$$hxm_\phi = \frac{16\frac{Z^4}{R^4}\ddot{F} + 8\frac{Z^3}{R^3}\ddot{D} - \frac{2Z}{R}\ddot{B}}{16Z^4/R^4} + \left[\, h + \frac{\frac{4Z^2}{R^2}\ddot{E} - \ddot{Q}}{4Z^2/R^2}\,\right] x - \frac{\ddot{A} + \frac{8Z^3}{R^3}\ddot{D}}{8Z^2/R^2}\, x^2$$
$$+ \frac{1}{6}\left(\ddot{C} - \frac{4Z^2}{R^2}\ddot{E}\right) x^3 + \frac{1}{24}\left(8\frac{Z^3}{R^3}\ddot{D} - 6\frac{Z}{R}\ddot{B}\right) x^4 \qquad (2.2.2.2.41)$$

4) $u_2 \leq x \leq 1$: regime 5-7

Again, the equilibrium equations, the boundary conditions (at $x = 1$) and the flow rules are exactly the same as those corresponding to the same interval, first stage of motion, of the medium pressure case.

Therefore equations (2.2.2.2.42)-(2.2.2.2.46) are applicable here.

5) Boundary Matching

There are 13 unknowns: u_o, u_1, u_2, $\dot{A}$, $\frac{2Z}{R}\dot{B}$, $\dot{C}$, $\dot{C}_o$, $\frac{4Z^2}{R^2}\dot{C}_1$, $\frac{8Z^3}{R^3}\ddot{D}$, $\frac{4Z^2}{R^2}\ddot{E}$, $\frac{16Z^4}{R^4}\ddot{F}$, $\dot{G}$, $\dot{K}$.

Noticing that the conditions of continuity of m_ϕ and s have been used in the determinations of the constants of integration and the continuity of n_ϕ is not in question at $x = u_o$, we have the following conditions to determine these unknowns:

a) Continuity of n_ϕ, s, m_ϕ at $x = u_1$ and $x = u_2$.

b) Continuity of $\dot{v}$, $\dot{w}$ and $\dot{w}'$ at $x = u_1$ and $x = u_2$.

The continuity of $\dot{w}'$ is required since at $x = u_1$ and $x = u_2$, $|m_\phi| < 1$ and since a discontinuity of $\dot{w}'$ is possible only when $|m_\phi| = 1$.

c) Continuity of $\dot{v}$ and $\dot{w}$ at $x = u_o$.

Here, at $x = u_o$, $m_\phi = 1$ and a discontinuity of $\dot{w}'$ is allowed; thus we have 14 conditions of continuity: 6 at each boundary $x = u_1$ and $x = u_2$ and 2 at $x = u_o$, resulting apparently in 14 equations for 13 unknowns, and the problem seems to be overspecified. Fortunately, the conditions of continuity of $\dot{w}$ and $\dot{v}$ at $x = u_o$ can be satisfied by one single equation and we have therefore 13 equations for 13 unknowns.

Solving and using the shallow shell approximation, we obtain the following results (Appendix A2)

$$\dot{C} = \frac{\dot{Q}}{1-u_o} \tag{2.2.2.3.14}$$

$$\frac{2Z}{R}\dot{B} = \dot{C} \tag{2.2.2.3.15}$$

$$\dot{K} = -\dot{C} \tag{2.2.2.3.16}$$

$$\dot{C}_o = -\dot{C} \tag{2.2.2.3.17}$$

$$\frac{4Z^2}{R^2}\dot{C}_1 = \dot{C}-\dot{Q} \tag{2.2.2.3.18}$$

$$\dot{A} = -\dot{C}(1 + \frac{2Z^2}{R^2} u_2{}^2) \qquad (2.2.2.3.19)$$

$$\ddot{C} = \frac{2(1-u_o)^2(1+2u_o)\ddot{Q}-12h}{(1-u_o)^3(1+3u_o)} \qquad (2.2.2.3.20)$$

$$\frac{\ddot{C}-2\ddot{Q}+12h}{3} - 2(\ddot{C}-\ddot{Q})u_o{}^2 + \frac{4}{3}(2\ddot{C}-\ddot{Q})u_o{}^3 - \ddot{C}u_o{}^4 =$$

$$\frac{4Z^2}{R^2}[\frac{1}{2}(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2)u_1^2 - \frac{3\ddot{C}-\ddot{Q}}{3}u_1^3 + \frac{5\ddot{C}}{12}u_1^4 + (\ddot{C}-\ddot{Q})u_o^2u_1 - \frac{2\ddot{C}}{3}u_o^3u_1]$$

$$(2.2.2.3.21)$$

$$\ddot{D}_1 = \frac{8Z^3}{R^3}\ddot{D}-\ddot{C} = \frac{2Z^2}{R^2}(\ddot{C}u_2{}^2 + 2\dot{C}\dot{u}_2u_2)$$

$$+ \frac{2Z^2/R^2}{(u_2-u_1)[2-(u_1+u_2)]}\left[\frac{\ddot{C}-2\ddot{Q}+12h}{3} - 2(\ddot{C}-\ddot{Q})u_o^2 + \frac{4}{3}(2\ddot{C}-\ddot{Q})u_o^3 - \ddot{C}u_o^4\right]$$

$$(2.2.2.3.22)$$

$$\ddot{E}_1 = \frac{4Z^2}{R^2}\ddot{E}-\ddot{Q} = \frac{(2Z^2/R^2)u_1}{(u_2-u_1)[2-(u_1+u_2)]}[\frac{\ddot{C}-2\ddot{Q}+12h}{3} - 2(\ddot{C}-\ddot{Q})u_o^2$$

$$+ \frac{4}{3}(2\ddot{C}-\ddot{Q})u_o^3-\ddot{C}u_o^4] - \frac{2Z^2}{R^2}[(\ddot{C}-\ddot{Q})u_o^2 - \frac{2}{3}\ddot{C}u_o^3]$$

$$(2.2.2.3.23)$$

$$\frac{\ddot{F}_1}{4Z^2/R^2} = \frac{\frac{16Z^4}{R^4}\ddot{F} + \frac{8Z^3}{R^3}\ddot{D}-\ddot{C}}{4Z^2/R^2} - \frac{(Z^2/R^2)u_1{}^2}{(u_2-u_1)[2-u_1+u_2)}[\frac{\ddot{C}-2\ddot{Q}+12h}{3}$$

$$- 2(\ddot{C}-\ddot{Q})u_o^2 + \frac{4}{3}(2\ddot{C}-\ddot{Q})u_o^3-\ddot{C}u_o^4] + \frac{Z^2}{R^2}[\frac{4}{3}(\ddot{C}-\ddot{Q})u_o^3-\ddot{C}u_o^4]$$

$$(2.2.2.3.24)$$

$$\ddot{G} = h - \frac{1}{2}\left[(\ddot{C}-\ddot{Q}) - \frac{2}{3}\ddot{C}u_o\right]u_o^2 - \frac{1}{2[2-(u_1+u_2)]}\left[\frac{\ddot{C}-2\ddot{Q}+12h}{3} - 2(\ddot{C}-\ddot{Q})u_o^2 + \frac{4}{3}(2\ddot{C}-\ddot{Q})u_o^3 - \ddot{C}u_o^4\right] \tag{2.2.2.3.25}$$

We observe that when $u_o = 0$, expressions (2.2.2.3.22)-(2.2.2.3.25) will reduce to expressions (2.2.2.2.55)-(2.2.2.2.58) of the medium pressure case.

The equations for u_o are:

$$(1-u_o)^2(1+u_o) = \frac{12h\tau}{\dot{Q}} \qquad (\tau > 0) \tag{2.2.2.3.26a}$$

$$(1-u_o)^2(1+u_o) = \frac{12h}{p_o-p_s+6h} \qquad (\tau = 0) \tag{2.2.2.3.26b}$$

We can also solve (2.2.2.3.26a) for u_o [21]:

$$u_o = \frac{1}{3} - \frac{4}{3}\cos\left(\frac{\pi+\phi}{3}\right) \tag{2.2.2.3.27}$$

$$\phi = \cos^{-1}\left\{\frac{81h}{4\dot{Q}} - 1\right\} \qquad (\tau > 0) \tag{2.2.2.3.28a}$$

$$\phi = \cos^{-1}\left\{\frac{81h}{4(p_o-p_s+6h)} - 1\right\} \qquad (\tau = 0) \tag{2.2.2.3.28b}$$

$\ddot{Q}$ and $\dot{Q}$ have been defined in (2.2.2.2.35) and (2.2.2.2.47) respectively.

With

$$F(\tau) = \frac{12h\tau}{Q} \qquad \text{and}$$

$$F(0) = \frac{12h}{p_o - p_s + 6h}$$

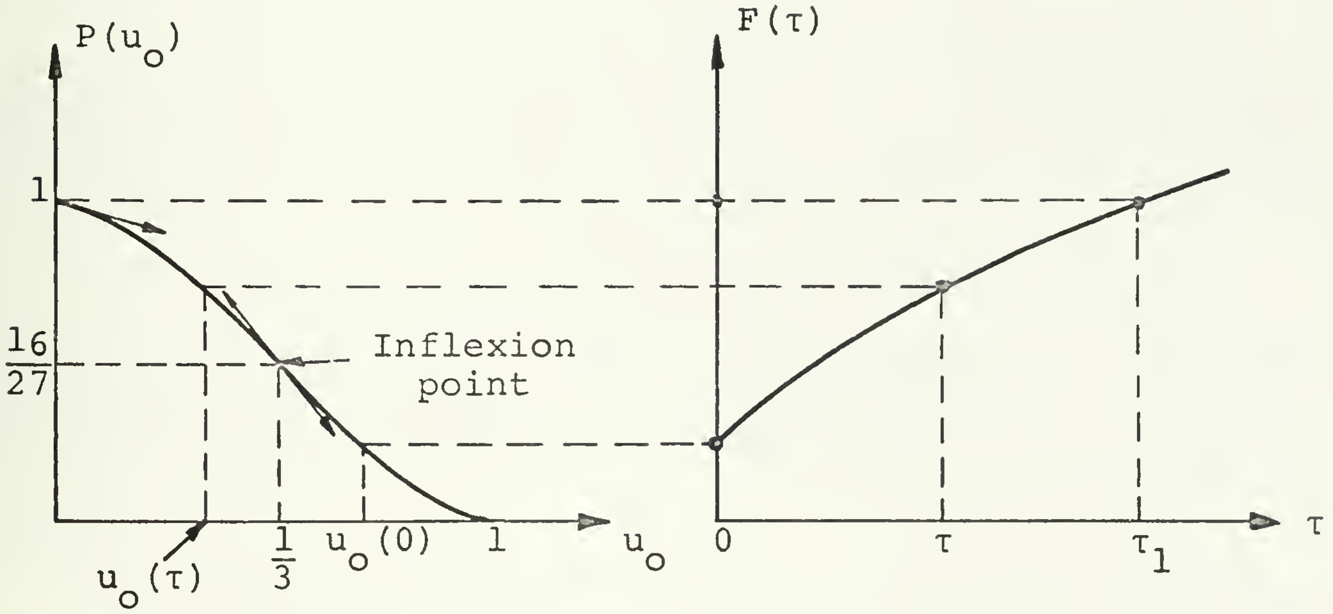
and

$$P(u_o) = (1-u_o)^2(1+u_o)$$

equations (2.2.2.3.26a,b) become:

$$P(u_o) = F(\tau) \qquad (2.2.2.3.29)$$

and the variations of u_o can be visualized in the following graphs.



Graphic Determination of $u_o(\tau)$

Fig. 15

At $\tau = \tau_o$, $u_o = 0$, with τ_o determined by:

$$F(\tau_o) = 1 \qquad \text{or}$$

$$p_o(1-e^{-\tau_o}) - p_s\tau_o - 6h\tau_o = 0 \qquad (2.2.2.3.30)$$

u_1 and u_2 are to be determined from:

$$u_2^2 = \frac{3(2+u_o)u_1^2 - 5u_1^3 - u_o^3}{3u_1} \qquad (2.2.2.3.31)$$

and

$$6(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2)(u_1+u_2) - 4(3\ddot{C}-\ddot{Q})(u_1^2+u_1u_2+u_2^2)$$

$$+ 5\ddot{C}(u_1+u_2)(u_1^2+u_2^2) + 12(\ddot{C}-\ddot{Q})u_o^2-8\ddot{C}u_o^3 = 0$$

$$(2.2.2.3.32)$$

We can see from (2.2.2.3.26b) that if $p_o-p_s = 6h$, $u_o(\tau=0) = 0$ and thus find by another way the upper limit of the pressure difference p_o-p_s for the medium pressure case.

At $\tau = 0$, the initial value u_{oo} of u_o is given by (2.2.2.3.27) and (2.2.2.3.28b). It is worthy to note that for h given, u_{oo} depends only on the pressure difference p_o-p_s.

To integrate (2.2.2.3.31) and (2.2.2.3.32) numerically, we note that (2.2.2.3.31) gives u_2 when u_1 and u_o are known. Therefore, it would be convenient if we can solve (2.2.2.31) and (2.2.2.3.32) for $\dot{u}_1$. To do it, we

differentiate (2.2.2.3.31) and using $\dot{u}_2 u_2$ from (2.2.2.3.32). After some algebraic manipulations, we obtain:

$$\begin{aligned}
\dot{u}_1 = {} & \frac{36hp_o[1-(1+\tau)e^{-\tau}]}{\dot{Q}^2} \times \frac{(u_1^2-u_o^2)u_1}{(1-u_o)(1+3u_o)[u_o^3+3(2+u_o)u_1^2-10u_1^3]} \\
& + \frac{1}{(1-u_o)^2(1+3u_o)[u_o^3+3(2+u_o)u_1^2-10u_1^3]\dot{Q}} \\
& \Big\{ -3[2(1-u_o)^2(1+2u_o)\ddot{Q}-12h]u_1^2u_2^2 \\
& + 2[(1-u_o)^2(5+10u_o+3u_o^2)\ddot{Q}-36h]\,\frac{(u_1^2+u_1u_2+u_2^2)u_1^2}{u_1+u_2} \\
& - \frac{5}{2}[2(1-u_o)^2(1+2u_o)\ddot{Q}-12h](u_1^2+u_2^2)u_1{}^2 \\
& - 2[(1-u_o)^2(3+2u_o+u_o^2)\ddot{Q} - 12h(3-2u_o)]\,\frac{u_o^2u_1^2}{u_1+u_2} \Big\}
\end{aligned}$$

(2.2.2.3.32a)

Thus u_1 and u_2 will be computed from (2.2.2.3.31) and (2.2.2.3.32a). To determine the initial values u_{10} and u_{20} of u_1 and u_2, we use the same argument that has been used in the medium pressure case to determine the initial value of g_o of g. Then we find that u_{10} and u_{20} are to be computed from the following equations:

$$- 3[2(1-u_{oo})^2(1+2u_{oo})\ddot{Q}_o-12h]u_{10}^2u_{20}^2$$

$$+ 2[(1-u_{oo})^2(5+10u_{oo}+3u_{oo}^2)\ddot{Q}_o - 36h]\frac{u_{10}^2+u_{10}u_{20}+u_{20}^2}{u_{10}+u_{20}}u_{10}^2$$

$$- \frac{5}{2}[2(1-u_{oo})^2(1+2u_{oo})\ddot{Q}_o-12h](u_{10}^2+u_{20}^2)u_{10}^2$$

$$- 2[(1-u_{oo})^2(3+2u_{oo}+u_{oo}^2)\ddot{Q}_o-12h(3-2u_{oo})]\frac{u_{oo}^2u_{10}^2}{u_{10}+u_{20}} = 0$$

and

$$u_{20} = [\frac{3(2+u_{oo})u_{10}^2-5u_{10}^3-u_{oo}^3}{3u_{10}}]^{1/2}$$

$\ddot{Q}_o$ is the value of $\ddot{Q}$ at $\tau = 0$, defined in (2.2.2.2.35).

With these initial values, the expression for $\dot{u}_1$ will have the form 0/0 at $\tau = 0$. Using the L'Hospital's rule, to find the true limit $\dot{u}_{10}$ of $\dot{u}_1$ at $\tau = 0$, we have found:

$$\dot{u}_{10} = \frac{p_o}{\ddot{Q}_o} \times \frac{H(u_{00},u_{10},u_{20})}{K(u_{00},u_{10},u_{20}} \quad ,$$

where with $S_{10} = u_{10}+u_{20}$ and $S_{20} = u_{10}^2 + u_{20}^2$, we have:

$$H(u_{oo},u_{10},u_{20}) = (1+2u_{oo}+3u_{oo}^2)S_{10}S_{20}(6+5u_{10}^2)$$

$$- 2(5+13u_{oo}+15u_{oo}^2+3u_{oo}^3) \times (S_{20}+u_{10}u_{20})u_{10}^2$$

$$+ 2(3+5u_{oo}^2+7u_{oo}^3-3u_{oo}^5)u_{10}^2 + (1-u_{oo}^2)(u_{10}^2-u_{oo}^2)$$

$$\times [(1+3u_{oo})(\frac{39}{4}u_{10}u_{20}+7u_{10}^2+\frac{5}{4}\frac{u_{10}^3}{u_{20}})-(2+7u_{oo}+3u_{oo}^2)(2u_{10}+\frac{u_{10}^2}{u_{20}})]$$

and

$$K(u_{oo},u_{10},u_{20}) = (1+3u_{oo})\Big\{(1+3u_{oo})\ [11u_{10}u_{20}^3 + \frac{33}{2}u_{10}^2u_{20}^2$$

$$+ 10u_{10}^3u_{20} + \frac{25}{2}u_{10}^4 + (2u_{oo}^3+3u_{10}u_{20}^2-5u_{10}^3)(\frac{13}{2}u_{20} + \frac{14}{3}u_{10}$$

$$+ \frac{5}{6}\frac{u_{10}^2}{u_{20}})] - 2(2+7u_{oo}+3u_{oo}^2)\ [4u_{10}^3+3u_{10}^2u_{20}+2u_{10}u_{20}^2$$

$$+ (2 + \frac{u_{10}}{u_{20}})\frac{2u_{oo}^3+3u_{10}u_{20}^2-5u_{10}^3}{3}] + 4u_{oo}^2(1+3u_{oo})u_{10}\Big\}$$

With these initial values, we can then go on to integrate numerically the system of equations made of (2.2.2.3.31) and (2.2.2.3.32a). (Appendix A4)

6. Final Results

With these unknowns determined, we have the following results:

a) $0 \leq x \leq u_o$: regime E-H

$$n_\theta = n_\phi = -1, \quad m_\theta = m_\phi = 1, \quad s = 0$$

$$\gamma\dot{v} = 0$$

From (2.2.2.3.4):

$$\gamma\dot{w} = \dot{Q}$$

b) $u_o \leq x \leq u_1$: regime E-7

$$n_\phi = n_\theta = -1, \quad m_\theta = 1$$

From (2.2.2.3.7):

$$xs = \frac{(1-u_0)^2(1+2u_o+3u_o^3)\ddot{Q}-12h}{2(1-u_o)^3(1+3u_o)}(x^2-u_o^2) - \frac{2(1-u_o)^2(1+2u_o)\ddot{Q}-12h}{3(1-u_o)^3(1+3u_o)}(x^3-u_o^3)$$

(2.2.2.3.33)

From (2.2.2.3.13):

$$hxm_\phi = hx + \frac{(1-u_o)^2(1+2u_o+3u_o^2)\ddot{Q}-12h}{6(1-u_o)^3(1+3u_o)}(x^3-3u_o^2x + 2u_o^3) - \frac{2(1-u_o)^2(1+2u_o)\ddot{Q}-12h}{12(1-u_o)^3(1+3u_o)}(x^4-4u_o^3x + 3u_o^4)$$

(2.2.2.3.34)

From (2.2.2.3.10):

$$\gamma \dot{w} = \frac{\dot{Q}}{1-u_o}(1-x) \qquad (2.2.2.3.35)$$

From (2.2.2.3.12):

$$\gamma \dot{v} = \frac{2Z}{R}\frac{\dot{Q}}{1-u_o}\frac{2x^3-3u_ox^2+u_o^3}{6x} \qquad (2.2.2.3.36a)$$

$$\gamma \dot{v} = \frac{2Z}{R}\frac{\dot{Q}}{1-u_o}\frac{(x-u_o)^2(2x+u_o)}{6x} \qquad (2.2.2.3.36b)$$

c) $u_1 \leq x \leq u_2$: regime 4-7

$$n_\theta = -1, \quad m_\theta = 1$$

From (2.2.2.2.39)

$$\begin{aligned} xn_\phi = \frac{Z}{R}\Bigg[& \frac{(3\ddot{C}-\ddot{Q})u_1u_2}{3(u_1+u_2)} - \frac{5\ddot{C}u_1u_2}{12} \\ & - \frac{2\ddot{C}u_o-3(\ddot{C}-\ddot{Q})}{3(u_1+u_2)}u_o^2\Bigg]u_1u_2 - x + \frac{Z}{R}\frac{2\ddot{C}u_o-3(\ddot{C}-\ddot{Q})}{3}u_o^2x \\ & - \frac{Z}{R}\left[\frac{(3\ddot{C}-\ddot{Q})(u_1^2+u_1u_2+u_2^2)}{3(u_1+u_2)} - \frac{5\ddot{C}}{12}(u_1^2+u_2^2) + \frac{2\ddot{C}u_o-3(\ddot{C}-\ddot{Q})}{3(u_1+u_2)}u_0^2\right]x^2 \\ & + \frac{1}{3}\frac{Z}{R}(3\ddot{C}-\ddot{Q})x^3 - \frac{5}{12}\frac{Z}{R}\ddot{C}x^4 \end{aligned}$$

(2.2.2.3.37a)

or:

$$xn_\phi = -x + \frac{Z}{R}(x-u_1)(u_2-x)\left\{\frac{2\ddot{C}u_o-3(\ddot{C}-\ddot{Q})}{3}\,\frac{u_o{}^2}{u_1+u_2} - \frac{3\ddot{C}-\ddot{Q}}{3}\,\frac{(u_1+u_2)x+u_1u_2}{u_1+u_2} + \frac{5}{12}\ddot{C}(x+u_1)(x+u_2)\right\}$$

(2.2.2.3.37b)

or finally:

$$n_\phi = -1 + \frac{Z/R}{(1-u_o)^3(1+3u_o)}\,\frac{(x-u_1)(u_2-x)}{x}$$

$$\left\{-\frac{(1-u_o)^2(3+2u_o+u_o^2)\ddot{Q}-12(3-2u_o)h}{3(u_1+u_2)}\,u_o^2\right.$$

$$-\frac{(1-u_o)^2(5+10u_o+3u_o^2)\ddot{Q}-36h}{3(u_1+u_2)}\,[(u_1+u_2)x+u_1u_2]$$

$$\left.+\frac{5}{12}\,[2(1-u_o)^2(1+2u_o)\ddot{Q}-12h](x+u_1)(x+u_2)\right\}$$

(2.2.2.3.38)

xs, hxm_ϕ and $\gamma\dot{w}$ remain the same as for $u_o \leq x \leq u_1$

From (2.2.2.2.37):

$$\gamma\dot{v} = \frac{Z}{R}\,\frac{\dot{Q}}{1-u_o}\,(-x^2+2x-u_2^2)$$

(2.2.2.3.39)

d) $u_2 \leq x \leq 1$: regime 5-7

$$n_\phi = -1, \quad m_\theta = 1$$

From (2.2.2.2.44):

$$n_\theta = -1 + \frac{2Z}{R}\left\{\frac{4}{3}\ddot{C}x^3 - \frac{3\ddot{C}-\ddot{Q}}{2}x^2 - \frac{1}{2}\,\frac{2\ddot{C}u_o-3(\ddot{C}-\ddot{Q})}{3}\,u_o^2\right\}$$

(2.2.2.3.40a)

and

$$n_\theta = -1 + \frac{2Z/R}{(1-u_o)^3(1+3u_o)}\left\{\frac{4}{3}[2(1-u_o)^2(1+2u_o)\ddot{Q}-12h]x^3\right.$$

$$- \frac{1}{2}[1-u_o)^2(5+10u_o+3u_o^2)\ddot{Q}-36h]x^2$$

$$\left. + \frac{1}{6}[(1-u_o)^2(3+2u_o+u_o^2)\ddot{Q}-12(3-2u_o)h]u_o^2\right\}$$

(2.2.2.3.40b)

xs, hxm_ϕ and $\gamma\dot{w}$ remain unchanged.

From (2.2.2.2.42):

$$\gamma\dot{v} = \frac{2Z}{R}\,\frac{\dot{Q}}{1-u_o}\,x(1-x) \qquad (2.2.2.3.41)$$

u_o is given by (2.2.2.3.27) and (2.2.2.3.28a,b).

u_1 and u_2 are solutions of (2.2.2.3.31) and (2.2.2.3.32a)

It has been verified that the solutions obtained satisfy the conditions of continuity at $x = u_o$, $x = u_1$, $x = u_2$ and the boundary conditions. At $x = u_o$, the slope $\gamma \dot{w}'$ is discontinuous and thus the curve $x = u_o$ is a hinge circle. It can be verified that across the hinge circle, the component $\dot{k}_\phi$ of the strain rate vector is positive, and therefore satisfying the normality requirement, since at $x = u_o$, $m_\phi = 1$.

It has been also verified that the solutions obtained satisfy the equations of equilibrium (2.1.2.1)-(2.1.2.3) within the limit of the shallow shell approximations.

At $\tau = \tau_o$ determined by (2.2.2.3.30), $u_o = 0$ and the first stage ends.

B. Second Stage of Motion: $\tau_o \leq \tau \leq \tau_1$

At $\tau = \tau_o$, $u_o = 0$ and the shell yields under 3 regimes as in the case of medium pressure.

When $u_o = 0$, then from (2.2.2.3.21), we have:

$$\ddot{C} = 2(\ddot{Q}-6h) = 2(p-p_s) + 0(\frac{Z^2}{R^2})$$

which is the same as (2.2.2.48) of the medium pressure case. Since in both medium and medium high pressure cases, the knowledge of $\ddot{C}$ will determine other unknowns involving stresses and acceleration, then these unknowns will have the same expressions and so are the stresses and accelerations.

To obtain the velocity, we have to integrate the accelerations from τ_o to τ. For $0 \leq x \leq u_1$, the acceleration $\gamma\ddot{w}$ comes from equation (2.2.2.2.5), which is:

$$\gamma\ddot{w} = \frac{Y'}{x} + p - \frac{4Z}{R} = \frac{(xs)'}{x} + p - ps + 6h$$

With s from (2.2.2.2.61), we have:

$$\gamma\ddot{w} = 2(p-p_s)(1-x)$$

Integrating from τ_o to τ, we have:

$$\gamma\dot{w}(\tau)-\gamma\dot{w}(\tau_o) = 2[p_o(e^{-\tau_o}-e^{-\tau}) - p_s(\tau-\tau_o)](1-x)$$

where $\gamma\dot{w}(\tau_o)$ is the velocity at $\tau = \tau_o$. From the continuity of $\gamma\dot{w}$ with respect to τ, we have:

$$\gamma\dot{w}(\tau_o) = \gamma\dot{w}(\tau_o^-)$$

and from (2.2.2.3.35),we have, with $u_o(\tau_o) = 0$:

$$\gamma\dot{w}(\tau_o^-) = \dot{Q}(\tau_o)(1-x)$$

and from the definition of $\dot{Q}$ in (2.2.2.2.47):

$$\gamma\dot{w}(\tau_o^-) = [p_o(1-e^{-\tau_o}) - p_s\tau_o + 6h\tau_o](1-x)$$

Thus, we have:

$$\gamma\dot{w}(\tau) = [p_o(1-e^{-\tau_o}) - p_s\tau_o + 6h\tau_o](1-x)$$

$$+ 2[p_o(e^{-\tau_o}-e^{-\tau}) - p_s(\tau-\tau_o)](1-x)$$

From equation (2.2.2.3.30), defining τ_o, we have:

$$6h\tau_o = p_o(1-e^{-\tau_o}) - p_s\tau_o$$

and thus:

$$\gamma\dot{w} = 2[p_o(1-e^{-\tau}) - p_s\tau](1-x)$$

which is the same as in the medium pressure case.

From the linearity of $\gamma\dot{w}$ and from the continuity of $\gamma\dot{w}$ and $\gamma\dot{w}'$ at $x = u_1$ and $x = u_2$, we obtain the same expression for the whole shell as in the medium pressure case.

For the meridional velocity, we observe from (2.2.2.3.14) that at τ_o, with $u_o = 0$, we have:

$$\dot{C}(\tau_o) = \dot{Q}(\tau_o)$$

and from a result just obtained above,

$$\dot{C}(\tau_o^-) = \dot{Q}(\tau_o) = 2[p_o(1-e^{-\tau_o}) - p_s\tau_o]$$

From:

$$\ddot{C} = 2(p-p_s) \quad \text{for } \tau \geq \tau_o\text{, we have:}$$

$$\dot{C}(\tau) - \dot{C}(\tau_o) = 2[p_o(e^{-\tau_o}-e^{-\tau}) - p_s(\tau-\tau_o)]$$

It can be seen that the condition of continuity of the velocities with respect to τ is reduced to that of $\dot{C}$. Thus

$C(\tau_o) = C(\tau_o^-)$ and we have:

$$\dot{C}(\tau) = 2[p_o(1-e^{-\tau_o}) - p_s\tau_o + p_o(e^{-\tau_o}-e^{-\tau}) - p_s(\tau-\tau_o)]$$

$$\dot{C}(\tau) = 2[p_o(1-e^{-\tau}) - p_s\tau]$$

which is the same as (2.2.2.2.49) of the medium pressure case.

Since all the unknowns involving in the different expressions of $\gamma\dot{v}$ are all expressible in terms of $\dot{C}$ and $\dot{Q}$, the former has been found to be the same and the latter remains unchanged from its initial definition in (2.2.2.2.47) we can conclude that the meridional velocities for different intervals $(0,u_1)$, (u_2,u_2), $(u_2,1)$ have the same expressions as those corresponding to the medium pressure case. Thus (2.2.2.2.61)-(2.2.2.2.68) are applicable here.

We observe also that all the stress expressions in this second stage can be obtained from the corresponding ones in the first stage by letting $u_o = 0$. Thus the stresses and accelerations are continuous with respect to τ as expected since the applied load is continuous.

u_1 and u_2 are to be determined from (2.2.2.2.59) and (2.2.2.2.60) with the values of u_1,u_2 at $\tau = \tau_o$ as initial conditions. Since at $\tau = \tau_o$,

$$\dot{C}(\tau_o) = \dot{Q}(\tau_o)$$

we have, from (2.2.2.2.59), at $\tau = \tau_o^+$:

$$u_2^2 = 2u_1 - \frac{5}{3} u_1^2$$

which is the same as (2.2.2.3.31) with $u_o = 0$, at $\tau = \tau_o^-$.

Also (2.2.2.3.32) is reduced to (2.2.2.2.60) when we make $u_o = 0$. Thus u_1, u_2 are also continuous at $\tau = \tau_o$.

At $\tau = \tau_1$ determined by (2.2.2.2.79), we have $u_1 = u_2 = 0$, and the second stage ends.

C. Third Stage of Motion.

After $\tau = \tau_1$, the whole shell collapses in one regime 5-7 as in the low pressure case. The solution is then the same as the second stage of motion of the medium pressure case.

Thus, $n_\phi = -1$, $m_\theta = 1$, and (2.2.2.1.9)-(2.2.2.1.13) are still valid.

D. Admissibility of the Solution

1. Kinematic Admissibility.

a) $0 \leq \tau \leq \tau_o$

i) $0 \leq x \leq u_o$: regime E-H

For kinematic admissibility in the interval,

we must have:

$$\dot{k}_\theta \geq 0, \quad \dot{k}_\phi \geq 0, \quad \dot{e}_\theta \leq 0, \quad \dot{e}_\phi \leq 0$$

With $\gamma\dot{w}$ from (2.2.2.3.4) and $\gamma\dot{v} = 0$, we have:

From (2.1.1.1): $\gamma\dot{e}_\theta = -\frac{2Z}{R}\dot{Q}$; since $\dot{Q}$ is positive for $0 \leq \tau \leq \tau_f$, we have $\dot{e}_\theta < 0$.

From (2.1.1.3), similarly, we have:

$$\gamma\dot{e}_\phi = \gamma(\dot{v}'-y''\dot{w}) = -\frac{2Z}{R}\dot{Q} < 0$$

From (2.1.1.2), we have:

$$\dot{k}_\theta = -h\,\frac{\dot{w}+y''\dot{v}}{x} = 0$$

Similarly, from (2.1.1.4):

$$\dot{k}_\phi = -h(\dot{w}'+y''\dot{v})' = 0$$

ii) $u_o \leq x \leq u_1$: regime E-7

For kinematic admissibility in this interval, we must have:

$$\dot{e}_\theta \leq 0, \quad \dot{e}_\phi \leq 0 \quad \dot{k}_\theta \geq 0$$

Using (2.1.1.1)-(2.1.1.4) again, and with $\gamma\dot{v}$ and $\gamma\dot{w}$ from (2.2.2.3.36a) and (2.2.2.3.35) respectively, we have:

$$\gamma\dot{e}_\theta = \frac{2Z}{R}\,\frac{\dot{Q}}{1-u_o}\,\frac{1}{6x^2}\left[\,8x^3 - 3(2+u_o)x^2 + u_o^3\,\right]$$

$$(u_o \leq x \leq u_1)$$

To have $\gamma\dot{e}_\theta \leq 0$, we must have:

$$P(x) = 8x^3-3(2+u_o)x^2+u_o^3 \leq 0 \qquad (u_o \leq x \leq u_1)$$

(2.2.2.3.42)

It can be shown that to have P(x) negative in the interval (u_o,u_1), it is necessary and sufficient to have either $P(u_o)$ or $P(u_1)$ negative, depending on the case.

From (2.2.2.3.42), we have: $P(u_o) = -6u_o^2(1-u_o)$ which is negative since $0 \leq u_o \leq 1$, and

$$P(u_1) = 8u_1^3-3(2+u_o)u_1^2+u_o^3$$

Using the definitionof u_2 in (2.2.2.3.31), and after some transformations, we have:

$$P(u_1) = -3u_1(u_2^2-u_1^2)$$

which is negative since $0 \leq u_1 \leq u_2$.

Therefore $\gamma\dot{e}_\theta$ is negative.

For $\gamma\dot{e}_\phi$, we have, from (2.2.2.3.35) and (2.2.2.3.36a):

$$\gamma\dot{e}_\phi = \frac{2Z}{R}\,\frac{\dot{Q}}{1-u_o}\,\frac{1}{6x^2}\,[10x^3-3(2+u_o)x^2-u_o^3]$$

To have $\gamma\dot{e}_\phi \leq 0$, we must have:

$$Q(x) = 10x^3-3(2+u_o)x^2-u_o^3 \leq 0 \qquad (u_o \leq x \leq u_1)$$

(2.2.2.3.43)

As before, it can be shown that for $\dot{Q}(x)$ to be negative in the interval (u_o, u_1), it is necessary and sufficient to have either $Q(u_o)$ or $Q(u_1)$ negative, depending on the case.

From (2.2.2.3.43), we have:

$$Q(u_o) = -6u_o^2(1-u_o) \leq 0$$

and

$$Q(u_1) = 10u_1^3-3(2+u_o)u_1^2-u_o^3$$

It has been found numerically that $Q(u_1)$ is negative.

Therefore $\gamma\dot{e}_\phi$ is negative. (Appendix A4)

For $\dot{k}_\theta$, we have, from (2.1.1.2) and with $\gamma\dot{w}$ and $\gamma\dot{v}$ from (2.2.2.3.35) and (2.2.2.3.36a) and using the shallow shell approximation:

$$\gamma\dot{k}_\theta \simeq \frac{h}{x}\frac{\dot{Q}}{1-u_o} > 0 \text{ since } \dot{Q} > 0 \text{ and } u_o < 1$$

With $\gamma\dot{k}_\theta > 0$, $\gamma\dot{e}_\theta < 0$, $\gamma\dot{e}_\phi < 0$, the solution is kinematically admissible in the interval (u_o, u_1).

iii) $u_1 \leq x \leq u_2$: regime 4-7

For kinematic admissibility in this interval, we must have $\dot{e}_\theta \leq 0$ and $\dot{k}_\theta \geq 0$.

As before, with the shallow shell approximation, we have, from (2.2.2.3.35) and (2.2.2.3.39):

$$\gamma\dot{k}_\theta \simeq \frac{h}{x}\,\frac{\dot{Q}}{1-u_o} \qquad \text{which is positive and}$$

$$\gamma\dot{e}_\theta = -\frac{Z}{R}\,\frac{\dot{Q}}{1-u_o}\,\frac{u_2^2-x^2}{x} \quad \text{which is negative}$$

since $u_1 \leq x \leq u_2$.

iv) $u_2 \leq x \leq 1$: regime 5-7:

For kinematic admissibility we must have:

$$\dot{k}_\theta \geq 0, \qquad \dot{e}_\phi \leq 0$$

With the shallow shell approximation, we have:

$$\gamma\dot{k}_\theta \simeq \frac{h}{x}\,\frac{\dot{Q}}{1-u_o}$$

which has been seen to be positive.

For $\dot{e}_\phi$, we have, from (2.2.2.3.35) and (2.2.2.3.41):

$$\gamma\dot{e}_\phi = -\frac{2Z}{R}\,\frac{\dot{Q}}{1-u_o}\,x$$

which is negative. Thus for $0 \leq \tau \leq \tau_o$, except for condition $Q(u_1) \leq 0$, we have been able to verify directly that the solution is kinematically admissible.

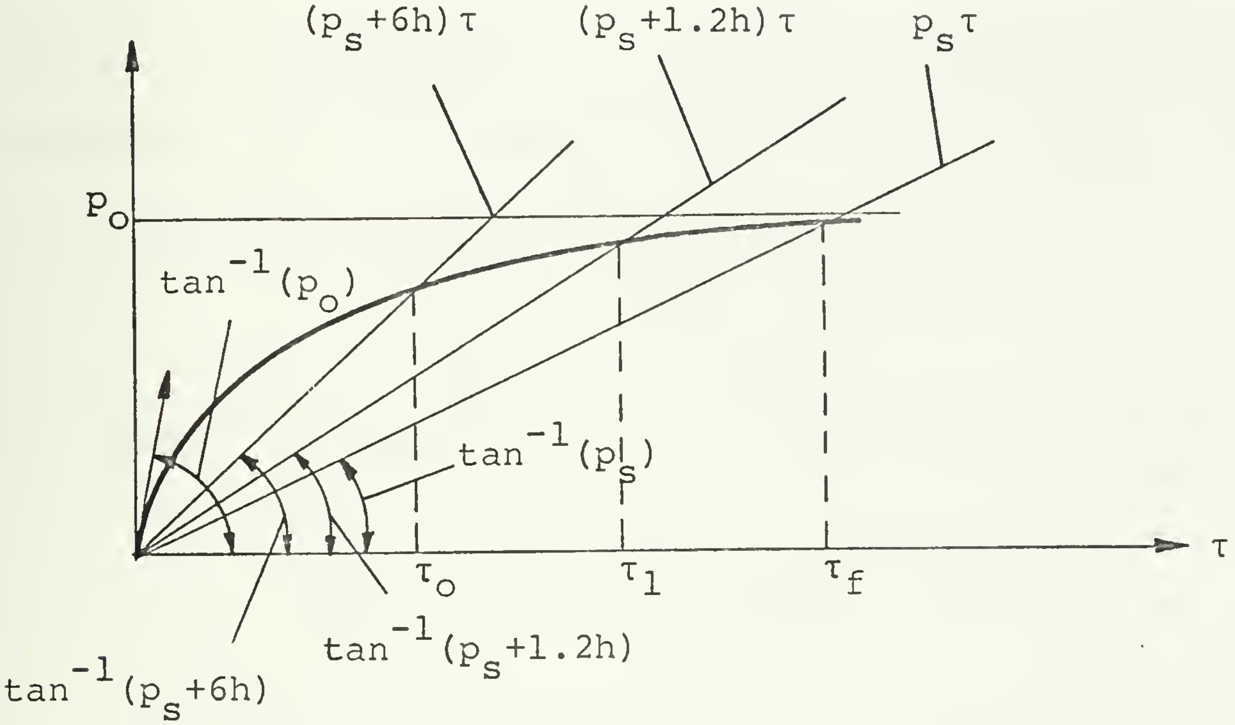
b) $\tau_o \leq \tau \leq \tau_1$

For this second stage of motion, it has been found from pages 81-84 that the solution is kinematically admissible if condition (2.2.2.2.89) is satisfied. This has been verified numerically and found that the solution is kinematically admissible. (Appendix A4)

c) $\tau_1 \leq \tau \leq \tau_f$

For this third stage of motion, the solution has been found to be kinematically admissible for $0 \leq \tau \leq \tau_f$ where τ_f is determined by (2.2.2.1.14) and which is the instant where the motion ceases.

The relative position of τ_o, τ_1, τ_f can be seen in the following graph:



Relative Positions of τ_o, τ_1, τ_f

Fig. 16

2. Static Admissibility

a) $0 \leq \tau \leq \tau_o$

i) m_ϕ

The conditions for admissibility are $0 \leq m_\phi \leq 1$.

For $0 \leq x \leq u_o$, $m_\phi = 1$ and is admissible.

For $u_o \leq x \leq 1$, we have from (2.2.2.3.13),

$$m_\phi = 1 - \frac{(x-u_o)^2}{12hx}\left\{\ddot{C}x^2+2[\ddot{Q}-(1-u_o)\ddot{C}]x + [4\ddot{Q}-(4-3u_o)\ddot{C}]u_o\right\} \tag{2.2.2.3.44}$$

In this form, we find:

$$m_\phi(u_o) = 1$$

$$m'_\phi(u_o) = 0 \text{ and } m''_\phi(u_o) \neq 0$$

Thus m_ϕ has an extremum at $x = u_o$. To be admissible, it is necessary that this extremum be a maximum or $m''_\phi(u_o) < 0$.

From the expression of m_ϕ, we have:

$$m''_\phi(u_o) = \frac{1}{h}[(1-u_o)\ddot{C}-\ddot{Q}] \tag{2.2.2.3.45}$$

From the expression of $\ddot{C}$ in (2.2.2.3.20) we have:

$$m''_\phi(u_o) = \frac{12}{(1-u_o)^2(1+3u_o)\dot{Q}}(\ddot{Q}\tau-\dot{Q})$$

With the expressions of $\ddot{Q}$ from (2.2.2.2.35) and of $\dot{Q}$ from (2.2.2.2.47), we have:

$$\ddot{Q}\tau-\dot{Q} = p_o[\tau e^{-\tau}-(1-e^{-\tau})]$$

Let us consider the function in the square brackets: $f(\tau) = \tau e^{-\tau}-(1-e^{-\tau})$. We have $df/d\tau = -\tau e^{-\tau}$. Thus $df/d\tau$ is negative for $\tau > 0$ and $f(\tau)$ is monotonically decreasing for $\tau > 0$. Since we have $f(0) = 0$, we conclude that $f(\tau)$ is negative for $\tau > 0$. Therefore $\ddot{Q}\tau-Q$ is negative for $\tau > 0$ and $m''_\phi(u_o) = 12/[(1-u_o)^2(1+3u_o)\dot{Q}] \times (\ddot{Q}\tau-\dot{Q})$ is negative as required, and $m_\phi(u_o) = 1$ is a maximum.

For m_ϕ to be admissible, we must also have: $0 \leq m_\phi \leq 1$ for $u_o \leq x \leq 1$. From the expression of m_ϕ in (2.2.2.3.44), we have:

$$m'_\phi = -\frac{x-u_o}{12hx^2}\left\{3\ddot{C}x^3+[3\ddot{C}u_o-4(\ddot{C}-\ddot{Q})](x^2+u_ox+u_o^2)\right\} \tag{2.2.2.3.46}$$

We can write m'_ϕ in the form:

$$m'_\phi = -\frac{x-u_o}{12hx^2}\left\{\ddot{C}(x-u_o)(3x^2+2u_ox+u_o^2) + 4[\ddot{Q}-(1-u_o)\ddot{C}](x^2+u_ox+u_o^2)\right\}$$

Let us consider the expression inside the curly brackets. In the second term, we will find $[\ddot{Q}-(1-u_o)\ddot{C}]$ positive if we compare it with (2.2.2.3.45) and note that we have found $m''_\phi(u_o) < 0$, and the second factor $x^2+u_ox+u_o^2$ is positive.

In the first term, the factors involving x are positive for $x \geq u_o$ and the coefficient $\ddot{C}$ has been found numerically to be positive, except for a few values of τ closed to τ_o for which $u_o \simeq 0$ and for a peak pressure close to the limit value p_{oL} determined by

$$p_{oL} = (p_s - 12h) e^{\tau_f}$$

But these few negative values of $\ddot{C}$ are also close to zero so that the expression inside the curly brackets is always positive. (Appendix A4)

Therefore m'_ϕ is negative for $0 \leq \tau \leq \tau_o$ and $u_o \leq x \leq 1$ and m_ϕ decreases monotonically from 1 to 0 as x varies from u_o to 1. Thus m_ϕ is admissible.

ii) n_ϕ

The conditions for admissibility are $-1 \leq n_\phi \leq 0$.

For $0 \leq x \leq u_1$, and $u_2 \leq x \leq 1$, we have $n_\phi = -1$

For $u_1 \leq x \leq u_2$, we have from (2.2.2.3.37b):

$$n_\phi = -1 + \frac{Z}{R} \frac{(x-u_1)(u_2-x)}{x} P(x) \qquad \text{with:}$$

$$P(x) = \frac{5}{12} \ddot{C} x^2 + [\frac{5}{12} \ddot{C}(u_1+u_2) - \frac{3\ddot{C}-\ddot{Q}}{3}] x$$

$$+ [\frac{5}{12}\ddot{C}(u_1+u_2) - \frac{3\ddot{C}-\ddot{Q}}{3}] \frac{u_1 u_2}{u_1+u_2} + [\frac{2}{3}\ddot{C}u_o - (\ddot{C}-\ddot{Q})] \frac{u_o^2}{u_1+u_2}$$

and the necessary condition is that P(x) be positive for $u_1 \le x \le u_2$.

As in the case of medium pressure, we have found numerically that u_1 and u_2 are both on the same side of the midpoint S/2 where S is the sum of the root of the equation

$$P(x) = 0$$

and that $0 < P(u_1) < P(u_2)$

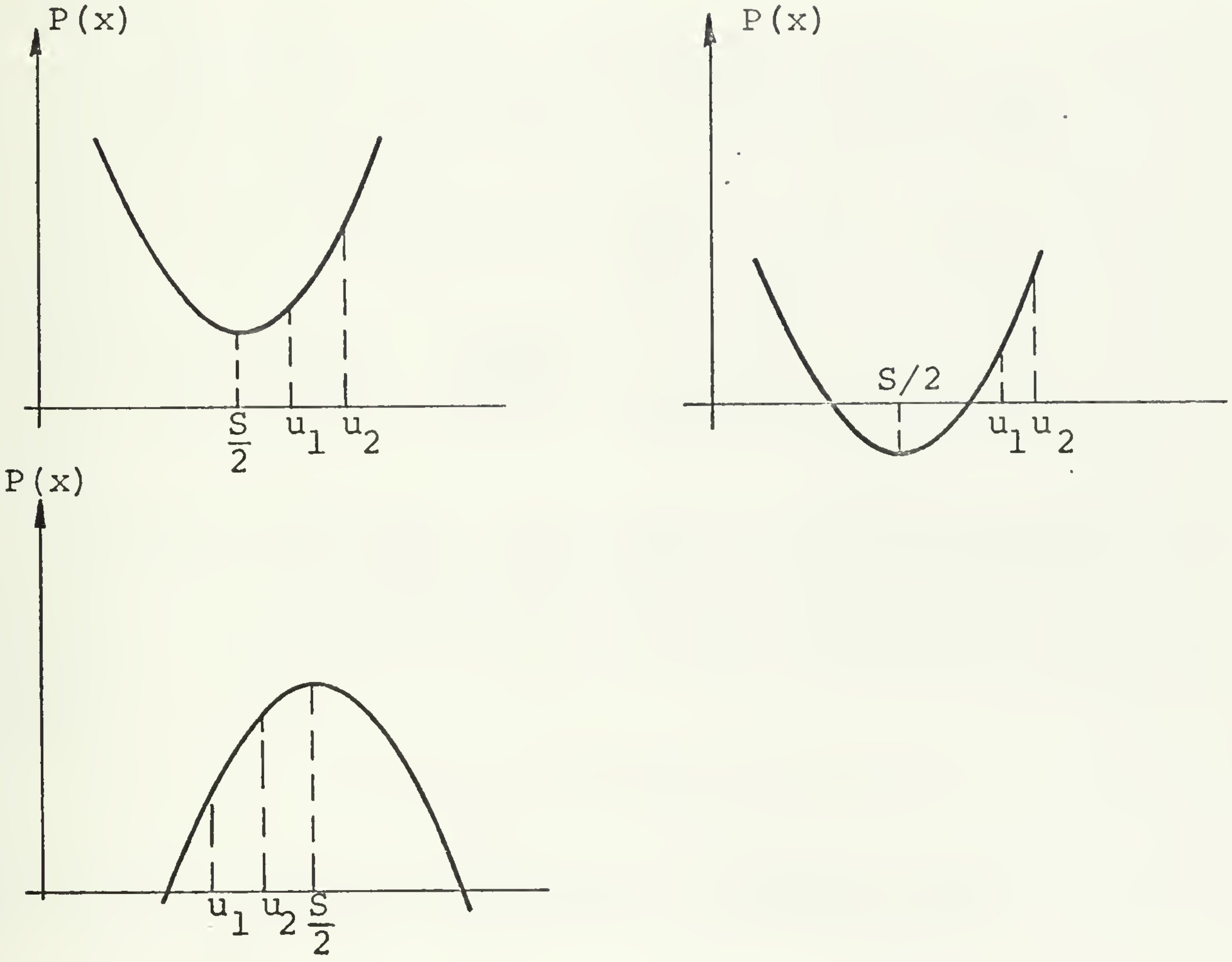


Fig. 13 a,b,c repeated

The maximum value of $P(u_2)$ is about 6h. An analysis analogous to page 95 allows us to write

$$(n_\phi)_{max} \leq -1 + \frac{Z}{R}[P(x)]_{max} \leq -1 + 6h\frac{Z}{R}$$

From this, we can see that n_ϕ is always negative, and therefore n_ϕ is admissible.

iii) n_θ

The conditions for admissibility are: $-1 \leq n_\theta \leq 0$.
For $0 \leq x \leq u_2$, $n_\theta = -1$ and is admissible.
For $u_2 \leq x \leq 1$, we have from (2.2.2.3.40a):

$$n_\theta = -1 + \frac{2Z}{R} R(x), \quad \text{with}$$

$$R(x) = \frac{4}{3}\ddot{C}x^3 - \frac{3\ddot{C}-\ddot{Q}}{2}x^2 + \frac{1}{2}[(\ddot{C}-\ddot{Q}) - \frac{2}{3}\ddot{C}u_o]u_o^2$$

It has been found that R(x) is always positive for $u_2 \leq x \leq 1$ and its maximum value is about 6h so that we have:

$$-1 \leq n_\theta \leq 0 \text{ for } u_2 \leq x \leq 1$$

and therefore n_θ is admissible. (Appendix A4)

b) $\tau_o \leq \tau \leq \tau_1$

i) n_ϕ

The conditions for admissibility are: $-1 \leq n_\phi \leq 0$.

The discussion is the same as in the case of medium

pressure and the numerical results obtained are the same, as far as the signs are concerned, that is, (DD1) = $u_1 - S/2$ and (DD2) = $u_2 - S/2$ have the same signs, $0 < P(u_1) < P(u_2)$, and the maximum value of $P(u_2)$ is of the order of 10h so that (Appendix A4):

$$(n_\phi)_{max} \le -1 + \frac{5}{6}\,\frac{Z}{R}\,\left[\frac{(x-u_1)(u_2-x)}{x}\right]_{max} \times [P(x)]_{max}$$

which is

$$(n_\phi)_{max} \le -1 + \frac{25}{3}\,\frac{Z}{R}\,h$$

Since h is of the order of a few per cent and Z/R is smaller than 1, we see that $(n_\phi)_{max}$ is still negative and the solution is admissible.

ii) m_ϕ

The conditions for admissibility are: $0 \le m_\phi \le 1$.

It has been found that m_ϕ is admissible (pages 51 and 52) if:

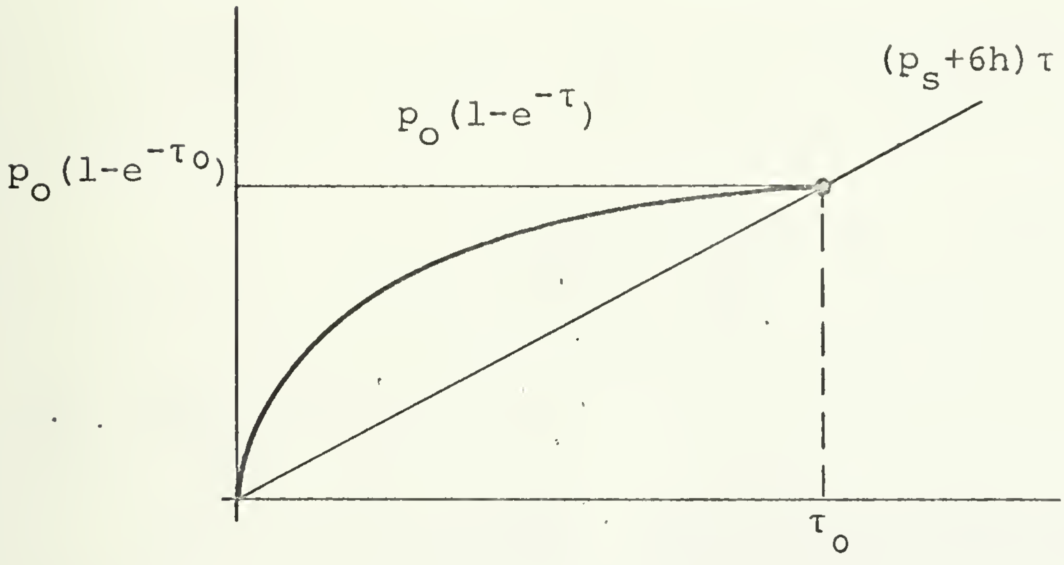
$$p - p_s \le 6h \quad \text{(AM1)} \qquad \text{and}$$

$$12h + p - p_s \ge 0 \quad \text{(AM2)}$$

Since $p = p_o e^{-\tau}$ decreases monotonically, condition (AM1) is satisfied for $\tau > \tau_o$ if it is at $\tau = \tau_o$.

Thus, we must have:

$$p_o e^{-\tau_o} - p_s \leq 6h$$



Graphic Determination of τ_o

Fig. 17

where τ_o is the root of equation (2.2.2.3.30).

As before, we note that the function $p_o(1-e^{-\tau})$ is continuous whose derivative is $p_o e^{-\tau}$. From the theorem of the mean, we have a value τ_m of τ in the interval $(0,\tau_o)$ such that:

$$\frac{p_o(1-e^{-\tau_o})}{\tau_o} = p_o e^{-\tau_m} \qquad , \qquad 0 < \tau_m < \tau_o$$

From (2.2.2.3.30), we have:

$$p_o(1-e^{-\tau_o}) = (p_s+6h)\tau_o$$

Therefore:

$$p_o e^{-\tau_m} = (p_s + 6h)$$

Since $\tau_m < \tau_o$ and $p_o e^{-\tau}$ decreases monotonically, we have:

$$p_o e^{-\tau_o} < p_s + 6h \quad \text{or:}$$

$$p_o e^{-\tau_o} - p_s < 6h$$

and (AM1) is satisfied.

For condition (AM2), again since p decreases monotonically, we see that it is satisfied for $\tau_o < \tau < \tau_f$ if it is at τ_f. Or:

$$(AM2) \equiv 12h + p_o e^{-\tau_f} - p_s \geq 0$$

where τ_f is the instant when the motion ceases and is determined by (2.2.2.1.14).

A numerical approach combining equation (2.2.2.1.14) with equation

$$12h + p_o e^{-\tau_f} - p_s = 0$$

will allow us to find τ_f and the upper limit p_{oL} of p_o for which (AM2) = 0. The value of p_{oL} depends on the depth of the shell. (Graph G.9)

iii) n_θ

From the study in the medium pressure case in pages 96-97 we can conclude that n_θ is admissible if condition (AD2) in (2.2.2.2.95) is satisfied. This has been done numerically and n_θ has been found to be admissible.

c) $\tau_1 \leq \tau \leq \tau_f$

For this stage of motion, we have $n_\phi = -1$, $m_\theta = 1$ and we have to examine m_ϕ and n_θ.

From a result in page 101 we have:

$$p_o e^{-\tau_1} - p_s < 1.2h$$

so that $n_\theta = -1$ has a minimum at $x = 0$. The other conditions remain the same, and we can conclude that the solution is admissible in this stage of motion if condition (AM2) is satisfied.

3. Conclusion

The solution has been found to be kinematically and statically admissible and is therefore an exact solution according to the yield surface which has been selected.

E. Final Displacement

1. Normal Displacement

a) $u_{oo} \leq x \leq 1$, where u_{oo} is the value of u_o

at $\tau = 0$, we have:

$$\gamma w_f = \int_0^{\tau_o} \gamma\dot{w}d\tau + \int_{\tau_o}^{\tau_f} \gamma\dot{w}d\tau$$

From (2.2.2.3.35) and (2.2.2.2.63), we have:

$$\gamma w_f = (1-x) \left[\int_0^{\tau_o} \frac{\dot{Q}}{1-u_o} d\tau + 2(p_o-p_s)\tau_f - 2(p_o-p_s-6h)\tau_o - p_s(\tau_f^2-\tau_o^2) \right] \tag{2.2.2.3.47a}$$

b) $0 \leq x \leq u_{oo}$

If τ_{ox} is the value of τ for which $u_o(\tau_{ox}) = x$, then we have:

$$\gamma w_f = \int_0^{\tau_{ox}} \gamma\dot{w}d\tau + \int_{\tau_{ox}}^{\tau_o} \gamma\dot{w}d\tau + \int_{\tau_o}^{\tau_f} \gamma\dot{w}d\tau$$

From the expressions of $\gamma\dot{w}$ in (2.2.2.3.4), (2.2.2.3.35) and (2.2.2.2.63), we have:

$$\gamma w_f = \int_0^{\tau_{ox}} \dot{Q}d\tau + (1-x) \left[\int_{\tau_{ox}}^{\tau_o} \frac{\dot{Q}}{1-u_o} d\tau + 2(p_o-p_s)\tau_f - 2(p_o-p_s-6h)\tau_o - p_s(\tau_f^2-\tau_o^2) \right] \tag{2.2.2.3.47b}$$

2. Tangential Component

Let $u_{10} = u_1(0)$, $u_{20} = u_2(0)$ and τ_{ox}, τ_{1x} and τ_{2x} determined by:

$$u_o(\tau_{ox}) = x$$

$$u_1(\tau_{1x}) = x$$

$$u_2(\tau_{2x}) = x$$

If all these quantities exist for a point x, we will have: $\tau_{ox} < \tau_{1x} < \tau_{2x}$.

a) $u_{20} \leq x \leq 1$

Then, none of the quantities τ_{ox}, τ_{1x}, τ_{2x} exists. We have:

$$\gamma v_f = \int_0^{\tau_o} \gamma\dot{v}d\tau + \int_{\tau_o}^{\tau_f} \gamma\dot{v}d\tau$$

From (2.2.2.3.41) and (2.2.2.2.64), we have:

$$\gamma v_f = \frac{4Z}{R}x(1-x)\left[\int_0^{\tau_o}\frac{\dot{Q}}{1-u_o}d\tau + (p_o-p_s)\tau_f - (p_o-p_s-6h)\tau_o - \frac{1}{2}p_s(\tau_f^2-\tau_o^2)\right]$$

(2.2.2.3.48a)

b) $u_{10} \leq x \leq u_{20}$

i) If x is such that $\tau_{2x} < \tau_o$, then we have:

$$\gamma v_f = \int_0^{\tau_{2x}} \gamma\dot{v}d\tau + \int_{\tau_{2x}}^{\tau_o} \gamma\dot{v}d\tau + \int_{\tau_o}^{\tau_f} \gamma\dot{v}d\tau$$

From (2.2.2.3.39), (2.2.2.3.41) and (2.2.2.2.68), we have:

$$\gamma v_f = \frac{Z}{R}\left[\int_0^{\tau_{2x}} \frac{\dot{Q}}{1-u_o}(-u_2^2+2x-x^2)d\tau + \int_{\tau_{2x}}^{\tau_o} \frac{2\dot{Q}}{1-u_o}x(1-x)d\tau\right]$$

$$+ \frac{4Z}{R}x(1-x)\ [(p_o-p_s)\tau_f - (p_o-p_s-6h)\tau_o - \frac{1}{2}p_s(\tau_f^2-\tau_o^2)]$$

(2.2.2.3.48b)

ii) If x is such that $\tau_o < \tau_{2x}$, we have:

$$\gamma v_f = \int_0^{\tau_o} \gamma\dot{v}d\tau + \int_{\tau_o}^{\tau_{2x}} \gamma\dot{v}d\tau + \int_{\tau_{2x}}^{\tau_f} \gamma\dot{v}d\tau$$

From (2.2.2.3.39), (2.2.2.2.66) and (2.2.2.2.68), we have:

$$\gamma v_f = \frac{Z}{R}\int_0^{\tau_o} \frac{\dot{Q}}{1-u_o}(-u_2^2+2x-x^2)\ d\tau$$

$$+ \frac{2Z}{R}\int_{\tau_o}^{\tau_{ox}} [p_o(1-e^{-\tau})-p_s\tau](-u_2^2+2x-x^2)d\tau$$

$$+ \frac{4Z}{R}x(1-x)\ [(p_o-p_s)\tau_f + p_o\tau_{2x} - p_o(1-e^{-\tau_{2x}})$$

$$- (1/2)p_s(\tau_f^2-\tau_{2x}^2)]$$ (2.2.2.3.48c)

c) Conclusion

From these examples, we can see that for other values of x such that $x \leq u_{10}$, the cases to be studied become numerous with the coming of the value of τ_1 into the problem. This lengthy analysis will be tedious, without much interest, since in all these cases, a numerical approach has to be used to compute the actual values. Moreover, the tangential displacement is of the order $0(Z/R)$ compared to the normal displacement.

The most important displacement is that of the apex of the shell, which has been determined in the following part and has been carried out numerically.

F. Maximum Central Deflection and Total Energy Absorbed.

1. Maximum Central Deflection

The maximum central deflection γw_o is given by:

$$\gamma\, w_o = \int_0^{\tau_f} \gamma \dot{w}_o \, d\tau$$

From (2.2.2.3.4) for $0 \leq \tau \leq \tau_o$ and, from the expression of $\gamma \dot{w}$ for $\tau_o \leq \tau \leq \tau_f$, we have:

$$\gamma \dot{w}_o = p_o(1-e^{-\tau}) - p_s\tau + 6h\tau \qquad (0 \leq \tau \leq \tau_o)$$

and, from page 126:

$$\gamma\dot{w}_o = 2[p_o(1-e^{-\tau})-p_s\tau] \qquad (\tau_o \le \tau \le \tau_f)$$

With these results, we have:

$$\gamma\dot{w}_o = \int_0^{\tau_o} \gamma\dot{w}_o d\tau + \int_{\tau_o}^{\tau_f} \gamma\dot{w}_o d\tau$$

$$\gamma w_o = 2[(p_o-p_s)\tau_f - \tfrac{1}{2}p_s\tau_f^2] + (p_s+6h)(1+\tfrac{1}{2}\tau_o)\tau_o - p_o\tau_o \qquad (2.2.2.3.49)$$

This result can also be obtained from (2.2.2.3.46b) with $x = 0$ and $\tau_{ox} = \tau_o$.

2. Total Energy Absorbed

From a result page 58, we have:

$$E_{abs} = 2\pi R^2 N_o \int_0^{\tau_f}\int_0^1 p\dot{w}x dx\, d\tau$$

$$E_{abs} = 2\pi R^2 N_o \left[\int_0^{\tau_o}\int_0^1 pwxdxd\tau + \int_{\tau_o}^{\tau_f}\int_0^1 p\dot{w}xdxd\tau \right]$$

For $0 \le \tau \le \tau_o$, we have, from (2.2.2.3.4):

$$\dot{w} = \dot{Q} \qquad (0 \le x \le u_o)$$

and from (2.2.2.3.35):

$$\gamma\dot{w} = \frac{\dot{Q}}{1-u_o}(1-x) \qquad (u_o \le x \le 1)$$

For $\tau_o \leq \tau \leq \tau_f$, we have from (2.2.2.2.63):

$$\gamma\dot{w} = 2[p_o(1-e^{-\tau}) - p_s\tau]\,(1-x) \qquad (0 \leq x \leq 1)$$

With these results, we have:

$$E_{abs} = 2\pi R^2 N_o \left\{ \int_0^{\tau_o} \left[\int_0^{u_o} p\dot{w}x\,dx + \int_{u_o}^{1} p\dot{w}x\,dx \right] d\tau + \int_{\tau_o}^{\tau_f} \int_0^1 p\dot{w}x\,dx\,d\tau \right\}$$

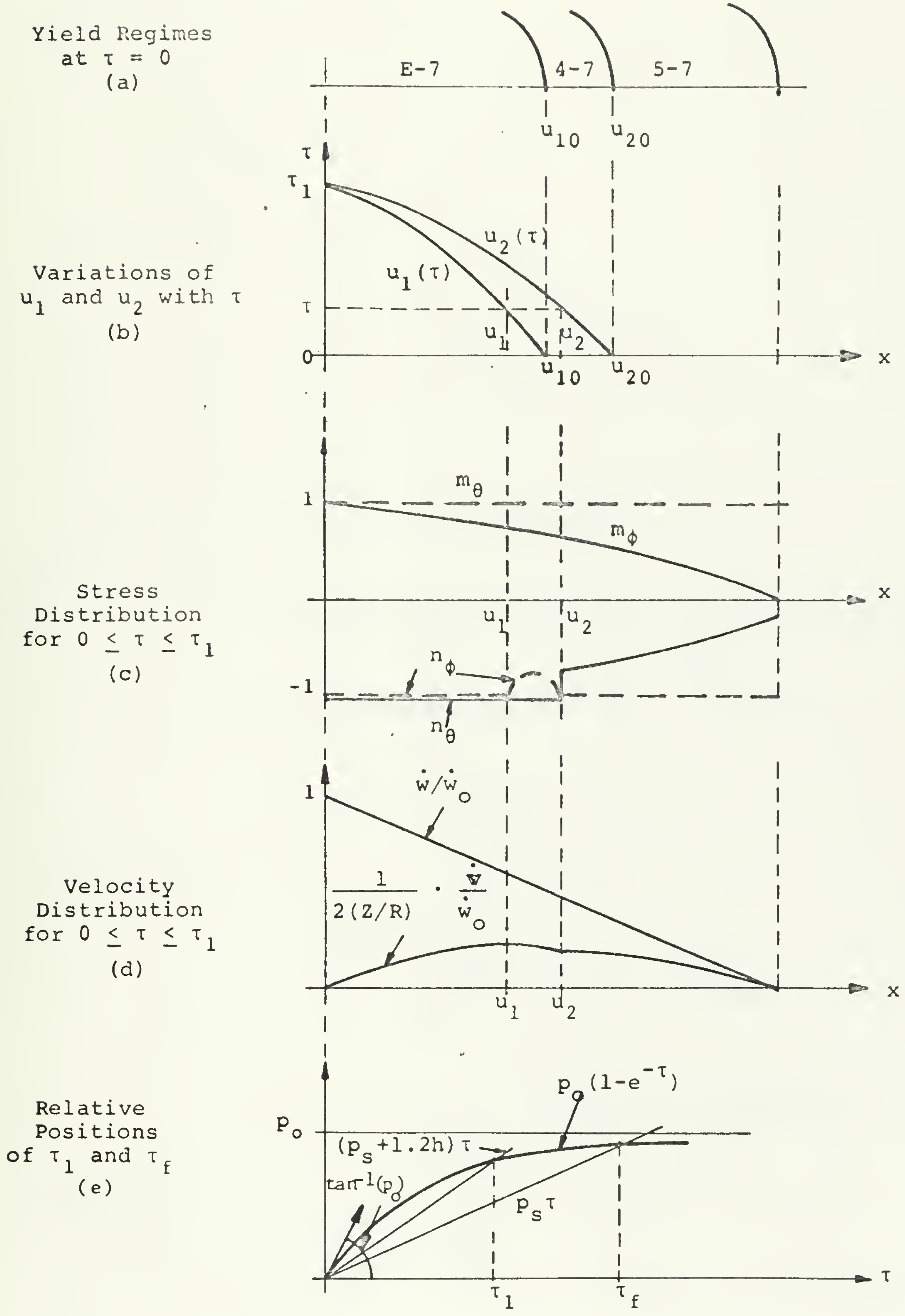
which gives, after integration with respect to the space variable:

$$E_{abs} = \frac{2\pi R^2 N_o}{3\gamma} \left\{ \int_0^{\tau_o} \frac{p_o}{2}\,[p_o(e^{-\tau}-e^{-2\tau})-(p_s-6h)\tau e^{-\tau}]\,(1+u_o+u_o^2)\,d\tau + \int_{\tau_o}^{\tau_f} p_o\,[p_o(e^{-\tau} - e^{-2\tau}) - p_s\tau e^{-\tau}]\,d\tau \right\}$$

Since u_o cannot be expressed explicitly in terms of τ, the first integral has to be evaluated numerically. The second integral can be performed, and we obtain:

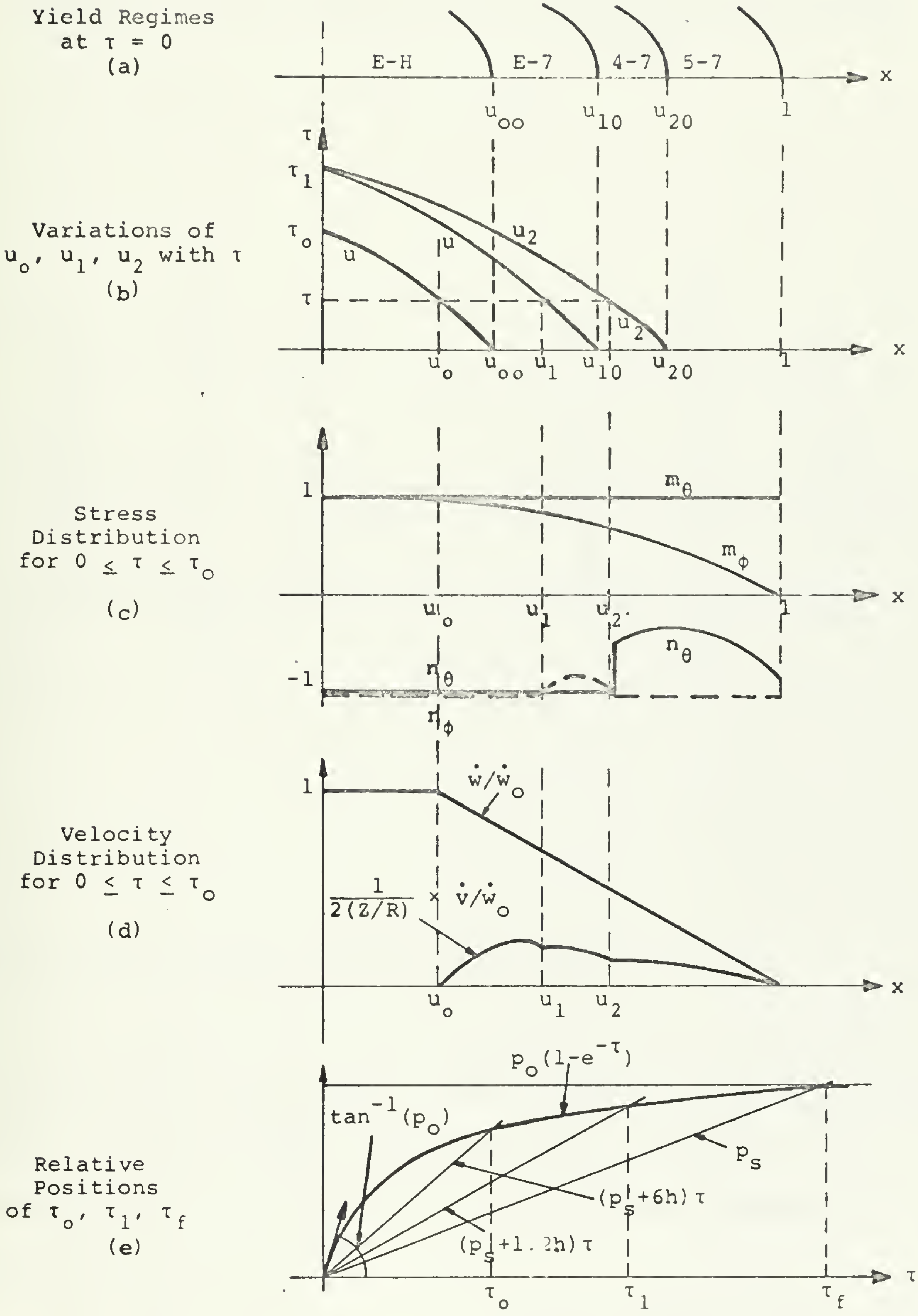
$$E_{abs} = \frac{2\pi R^2 N_o}{3\gamma} \left\{ \int_0^{\tau_o} \frac{p_o}{2}\,[p_o(e^{-\tau}-e^{-2\tau}) - (p_s-6h)\tau e^{-\tau}]\,(1+u_o+u_o^2)\,d\tau + \frac{1}{2}\,p_o^2\,[(1-e^{-\tau_f})^2 - (1-e^{-\tau_o})^2] - p_s p_o\,[(1+\tau_o)e^{-\tau_o}-(1+\tau_f)e^{-\tau_f}] \right\}$$

(2.2.2.3.50)



Medium Pressure - Shallow Shell - Two-Moment Limited Interaction Yield Surface

Fig. 18



Yield Regimes at $\tau = 0$ (a)

Variations of u_o, u_1, u_2 with τ (b)

Stress Distribution for $0 \leq \tau \leq \tau_o$ (c)

Velocity Distribution for $0 \leq \tau \leq \tau_o$ (d)

Relative Positions of τ_o, τ_1, τ_f (e)

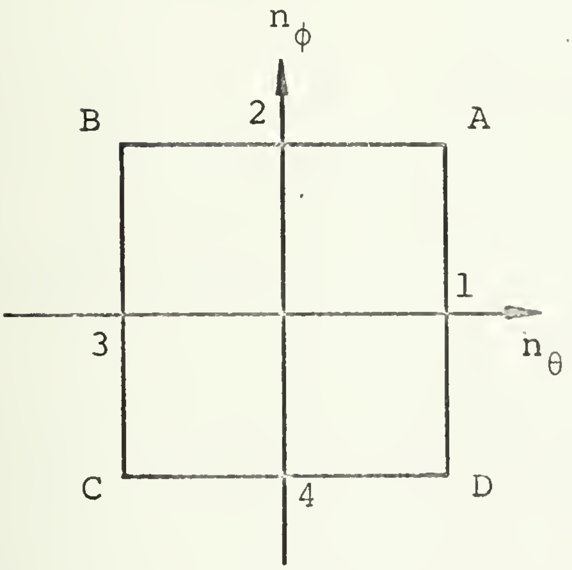
Medium High Pressure

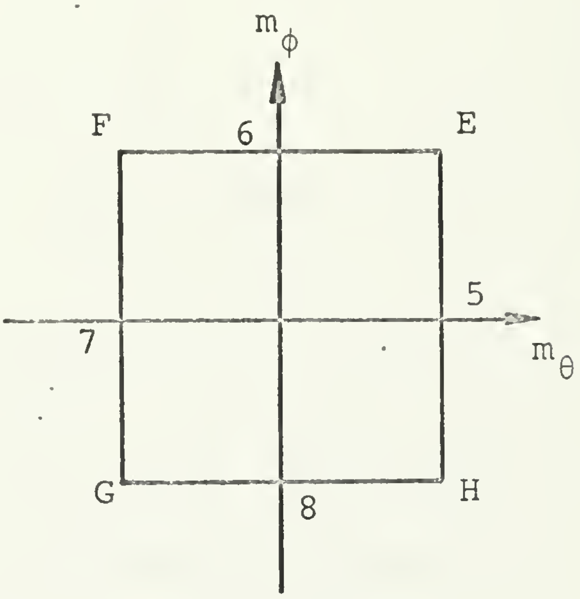
Shallow Shell - Two-Moment Limited Interaction Yield Surface

Fig. 19

2.3 Dynamic Plastic Response of Simply Supported Shallow Spherical Shells made of Rigid Plastic Material that Obeys the Uncoupled Square Yield Surface

2.3.1 Static Collapse Pressures





The yield regime for the whole shell is 4-5.

1. If the shell is simply supported, this will result in the same equations and the same boundary conditions as in section 2.2.1 for simply supported shell. Therefore the same results will apply to this case, and the collapse static pressure p_s is

a) From (2.2.1.27) for a shallow shell whose equation is:

$$y = \frac{Z}{R} x^n \quad , \quad n \geq 2:$$

$$p_s = 6h + \frac{6n}{n+1} \frac{Z}{R}$$

now that m_ϕ is allowed to be negative, the condition for admissibility becomes: $(m_\phi)_{min} > -1$, which will yield:

$$\frac{n(n-2)}{n+1} \frac{Z}{R} \leq (n+2)h$$

b) From (2.2.1.28) page for a shallow shell of second degree:

$$p_s = 6h + 4 \frac{Z}{R}$$

2. If the shell is clamped, expression (2.2.1.12) page for a shallow shell of second degree becomes:

$$hxm_\phi = hx - \frac{1}{6}(p - \frac{4Z}{R}) x^3$$

which yields:

$$m_\phi = 1 - \frac{1}{6h} (p - \frac{4Z}{R}) x^2$$

At $x = 1$, $m_\phi = -1$ or:

$$p = p_{sc} = 12h + \frac{4Z}{R} \qquad (2.3.1.1)$$

With this expression of p, the stresses are:

$$n_\phi = -1, \quad m_\theta = 1$$

$$s = -6hx$$

$$n_\theta = -1 + 12h \frac{Z}{R} x^2$$

$$m_\phi = 1 - 2x^2$$

The flow rules remain the same as in the simply supported case, and we have:

$$\dot{w} = \dot{w}_o(1 - x)$$

$$\dot{v} = \frac{2Z}{R}\,\dot{w}_o x(1 - x)$$

From these results, we can verify that the solution is statically and kinematically admissible and is therefore an exact solution.

2.3.2 Dynamic Response

Depending on the pressure difference $p_o - p_s$, the yield regimes are:

a) 4-5 for the whole shell, resulting in the same equations and the same results as in the low pressure case of section 2.2.2.1 and is valid for $0 \leq p_o - p_s \leq 1.2h$.

b) C-5 for $0 \leq x \leq u_1$, 3-5 for $u_1 \leq x \leq u_2$ and 4-5 for $u_2 \leq x \leq 1$, resulting in the same equations and the same results as in section 2.2.2.2 and is valid for $1.2h \leq p_o - p_s \leq 6.0h$.

c) C-E for $0 \leq x \leq u_o$, C-5 for $u_o \leq x \leq u_1$, 3-5 for $u_1 \leq x \leq u_2$ and 4-5 for $u_2 \leq x \leq 1$ resulting in the same equations and the same results as in section 2.2.2.3 and is valid for $6.0\ h < p_o - p_s \leq \lambda h$, where the value of λ depends on the depth of the shell and is not the same as in section 2.2.2.3, since the condition which m_ϕ

has to satisfy is now different: in this case m_ϕ is allowed to be negative. Therefore condition (2.2.2.1.19) which is:

$$p_o e^{-\tau} - p_s + 12h \geq 0$$

is now not necessary and has to be substituted by the one which expresses that the minimum value of m_ϕ at $x = x_m$ be larger than -1.

From (2.2.2.1.18) page , which is

$$x_m = \frac{2}{3} \frac{6h + (p_s - p)}{p_s - p}$$

we have:

$$(m_\phi)_{min} = 1 - \frac{6h + p_s - p}{18h} x_m^2$$

and we must have

$$1 - \frac{6h + p_s - p}{18h} x_m^2 \geq -1 \qquad \text{or}$$

$$\frac{6h + p_s - p}{18h} x_m^2 \leq 2$$

Since $x_m^2 \leq 1$, this condition is satisfied if we have:

$$\frac{6h + p_s - p}{18h} \leq 2 \qquad \text{or}$$

$$p_o e^{-\tau_f} - p_s + 30h \geq 0$$

This condition will be satisfied for $0 \leq \tau \leq \tau_f$ if it is at $\tau = \tau_f$. Therefore we must have:

$$p_o e^{-\tau_f} - p_s + 30h \geq 0 \qquad (2.3.2.1)$$

If we compare this condition with the one which expresses that n_θ be larger than -1 at $x = 1$ in (2.2.2.1.26) and which is:

$$p_o e^{-\tau_f} - p_s + 18h \geq 0 \ ,$$

we can see that condition (2.3.2.1) will be satisfied if condition (2.2.2.1.26) is.

Thus condition (2.2.2.1.19) page 43 in the case of two-moment limited-interaction yield surface is replaced by condition (2.2.2.1.26) for the case of uncoupled square yield surface. This later condition gives a limit value p_{oL} of p_o higher than that given by the former, and which can be found from the simultaneous system of equations:

$$\begin{cases} p_{oL}(1 - e^{-\tau_f}) - p_s \tau_f = 0 \\ p_{oL} e^{-\tau_f} - p_s + 18h = 0 \end{cases}$$

where τ_f and p_{oL} are the unknowns.

This system has been solved numerically and the values of p_{oL} computed for shells whose depth varies from 1 to 5 times their thickness.

2.4 Dynamic Plastic Response of Simply Supported Shallow Shells made of rigid-perfectly plastic material that obeys the uncoupled diamond yield surface.

2.4.1 Static Collapse Pressures

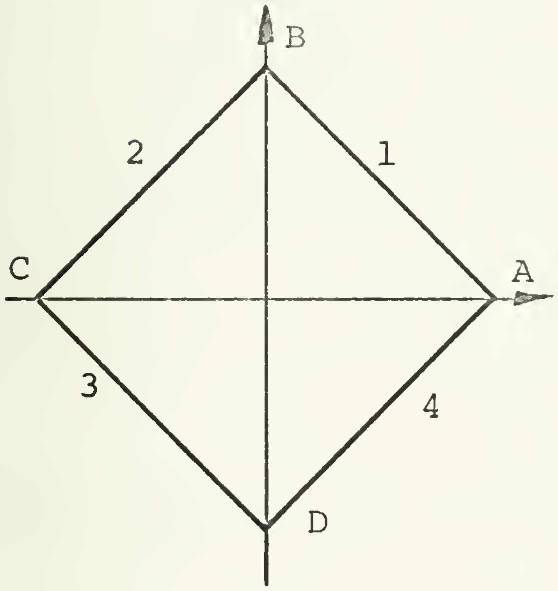
A. Simply Supported Edge

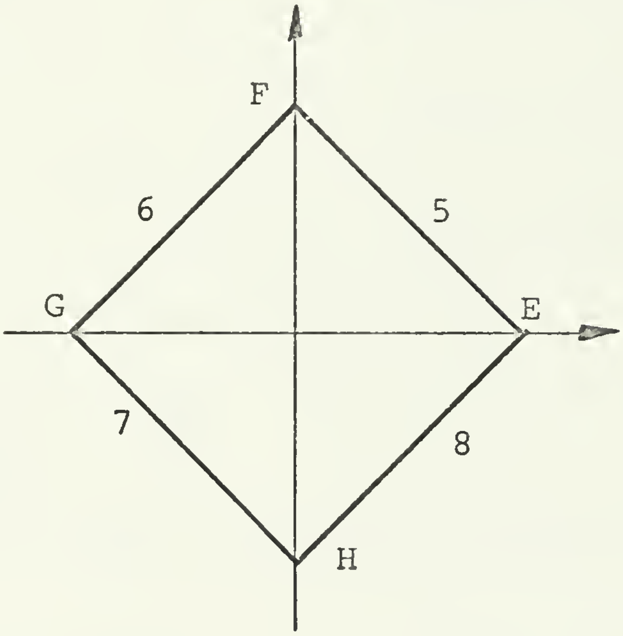
The equilibrium equations are the same as (2.2.1.1)-(2.2.1.3).

The boundary conditions are also the same, which are, for a simply supported edge:

$$x = 0: \quad s = 0, \; n_\theta = n_\phi, \; m_\theta = m_\phi$$

$$x = 1: \; m_\phi = 0 \qquad (2.2.1.4)$$





As in section 2.2.1, the examination of the boundary conditions will allow us to narrow the choices on yield regimes and to select the correct one which has been found to be made of faces 3 and 5.

For this yield regime, we have:

$$n_\theta + n_\phi = -1 \quad (2.4.1.1) \quad \text{(Table 2.1.3.3)}$$

$$m_\theta + m_\phi = 1 \quad (2.4.1.2) \quad \text{(Table 2.1.3.3)}$$

With (2.2.1.1)-(2.2.1.3) and (2.4.1.1), (2.4.1.2), we have the five equations needed to solve for the 5 unknowns stresses: s, n_θ, n_ϕ, m_θ, m_ϕ.

It has been found that:

$$s = -\frac{1}{2}(p - \frac{2Z}{R})x \qquad (2.4.1.3)$$

$$n_\phi = -\frac{1}{2} - \frac{1}{4}\frac{Z}{R}(p - \frac{2Z}{R})x^2 \qquad (2.4.1.4)$$

$$n_\theta = -\frac{1}{2} + \frac{1}{4}\frac{Z}{R}(p - \frac{2Z}{R})x^2 \qquad (2.4.1.5)$$

$$m_\phi = \frac{1}{2} - \frac{1}{8h}(p - \frac{2Z}{R})x^2 \qquad (2.4.1.6)$$

$$m_\theta = \frac{1}{2} + \frac{1}{8h}(p - \frac{2Z}{R})x^2 \qquad (2.4.1.7)$$

At x = 1, we have $m_\phi = 0$, or from (2.4.1.6)

$$0 = \frac{1}{2} - \frac{1}{8h}(p - \frac{2Z}{R})$$

From this equation, we have the static collapse pressure p_s:

$$p = p_s = 4h + \frac{2Z}{R} \qquad (2.4.1.8)$$

With this value of p, equations (2.4.1.3)-(2.4.1.7) become:

$$s = -2hx \qquad (2.4.1.9)$$

$$n_\theta = -(1/2) + h\frac{Z}{R}x^2 \qquad (2.4.1.11)$$

$$n_\phi = -\frac{1}{2} - h\frac{Z}{R}x^2 \qquad (2.4.1.10)$$

$$m_\phi = \frac{1}{2}(1-x^2) \qquad (2.4.1.12)$$

$$m_\theta = \frac{1}{2}(1+x^2) \qquad (2.4.1.13)$$

From these equations, and observing that h and Z/R are small and the product h(Z/R) is small compared to 1, we can see that the solution is statically admissible.

The flow rules for regime 3-5 are:

$$\dot{e}_\theta = \dot{e}_\phi \qquad (2.4.1.14)$$

$$\dot{k}_\theta = \dot{k}_\phi \qquad (2.4.1.15)$$

With $\dot{e}_\theta$, $\dot{k}_\theta$, $\dot{e}_\phi$, $\dot{k}_\phi$ from (2.1.1.1)-(2.1.1.4), equations (2.4.1.14) and (2.4.1.15) yield the following ones in terms of velocity components:

$$\dot{v} - y'\dot{w} = (\dot{v}' - y''\dot{w})x \qquad (2.4.1.16)$$

$$\dot{w}' + y''\dot{v} = (\dot{w}' + y''\dot{v})'\, x \qquad (2.4.1.17)$$

From these equations, and observing that for a shallow shell of second degree from (2.2.1.30), (2.2.1.31): $y' = y''x = \frac{2Z}{R}x$, we obtain:

$$\dot{v} = Ax \qquad (2.4.1.18)$$

$$\dot{w} = Bx^2 + \dot{w}_o \qquad (2.4.1.19)$$

Using the boundary condition of x = 1, we find A = 0 and $B = -\dot{w}_o$. Thus:

$$\dot{v} = 0 \qquad (2.4.1.20)$$

$$\dot{w} = \dot{w}_o(1-x^2) \qquad (2.4.1.21)$$

With these velocity expressions and from (2.1.1.1)-(2.1.1.4) with y' and y" from (2.2.1.30) and (2.2.1.31) we find:

$$\dot{e}_\theta = \dot{e}_\phi = - \frac{2Z}{R} \dot{w}_o (1-x)$$

$$\dot{k}_\theta = \dot{k}_\phi = 2h\dot{w}_o$$

Since $\dot{w}_o$ is positive and $x \leq 1$, we have:

$$\dot{e}_\theta = \dot{e}_\phi \leq 0, \quad \dot{k}_\theta = \dot{k}_\phi \geq 0$$

Thus the solution is also kinematically admissible. Therefore the solution obtained is an exact one for the yield surface selected.

B. Clamped Edge

If the shell is clamped at the edge $x = 1$, the boundary conditions at $x = 1$ become:

$x = 1, \quad \dot{w} = 0, \; \dot{w}' = 0$ or there will be a hinge

Since a hinge is likely to develop at the clamped edge, we assume the following yield regimes:

1. $0 \leq x \leq u$: regime 3-5

The system of equations for this regime is still made of (2.2.1.1)-(2.2.1.3) and (2.4.1.1), (2.4.1.2). Therefore the results (2.4.1.3)-2.4.1.7) remain valid.

The flow rules (2.4.1.14), (2.4.1.15) still apply, and therefore equations (2.4.1.18), (2.4.1.19) remain valid.

2. $u \leq x \leq 1$: regime 3-8

For this regime, the system of equations to determine the stresses are (2.2.1.1)-(2.2.1.3), (2.4.1.1) and the

equation corresponding to face 8 of the yield surface:

$$m_\theta - m_\phi = 1 \text{ (Table 2.1.3.3)} \qquad (2.4.1.22)$$

Since s, n_ϕ, n_θ are to be determined from (2.2.1.2), (2.2.1.3), (2.4.1.1) as before, equations (2.4.1.3)-(2.4.1.5) still apply. To determine m_ϕ and m_θ, we have (2.2.1.1) and (2.4.1.22). Solving this system with s from (2.4.1.3), we obtain:

$$m_\phi = \log \frac{x}{u} - \frac{1}{4h} (p - \frac{2Z}{R}) (x^2-u^2) \qquad (2.4.1.23)$$

$$m_\theta = 1 + \log \frac{x}{u} - \frac{1}{4h} (p - \frac{2Z}{R}) (x^2-u^2) \qquad (2.4.1.24)$$

where $m_\phi(u^+) = 0$ results from the fact that m_ϕ must be continuous at x = u.

The flow rule (2.4.1.14) still applies from which equation (2.4.1.18) remains valid.

Using the boundary condition at x = 1, we have:

$$A = 0 \text{ and } \dot{v} \equiv 0 \qquad (0 \leq x \leq 1) \qquad (2.4.1.25)$$

The flow rule corresponding to face 8 of the yield surface is (Table 2.1.3.3):

$$\dot{k}_\phi + \dot{k}_\theta = 0 \qquad (2.4.1.26)$$

With k_θ and k_ϕ from (2.1.1.2), (2.1.1.4), and with $\dot{v} = 0$, and the boundary condition: $\dot{w} = 0$ at x = 1, we have:

$$\dot{w} = C \log x \qquad (2.4.1.27)$$

3. Determination of the Unknowns

There are 4 unknowns: B, C, u, p, for which we have 4 equations:

i) Continuity of m_ϕ at $x = u$ expressed by the equations:

$$m_\phi(u^-) = m_\phi(u^+) = 0$$

or, from (2.4.1.6)

$$0 = \frac{1}{2} - \frac{1}{8h}(p - \frac{2Z}{R})u^2 \qquad (2.4.1.28)$$

ii) Continuity of $\dot{w}$ at $x = u$, which, from (2.4.1.19), gives:

$$Bu^2 + \dot{w}_o = C \log u \qquad (2.4.1.29)$$

iii) Continuity of $\dot{w}'$ at $x = u$, since $\dot{w}'$ can be discontinuous only for $|m_\phi| = 1$, which is not the case here. Differentiating (2.4.1.19) and (2.4.1.27) with respect to x, we obtain:

$$2Bu = \frac{C}{u} \qquad (2.4.1.30)$$

iv) The boundary condition at $x = 1$: for $\dot{w}'$ to be discontinuous at $x = 1$, we must have $m_\phi = -1$. Then from (2.4.1.23) we have:

$$-1 = \log \frac{1}{u} - \frac{1}{4h}(p - \frac{2Z}{R})(1-u^2) \qquad (2.4.1.31)$$

Solving (2.4.1.29) and (2.4.1.30) for B and C, we obtain:

$$B = -\frac{\dot{w}_o}{u^2(1-2\log u)} \qquad (2.4.1.32)$$

$$C = -\frac{2\dot{w}_o}{1-2\log u} \qquad (2.4.1.33)$$

Solving (2.4.1.28) and (2.4.1.31) for u and p, we obtain:

$$p = p_{sc} = \frac{4h}{u^2} + \frac{2Z}{R} \qquad (2.4.1.34)$$

and

$$\frac{1}{u^2} = 2 - \log u \qquad (2.4.1.35)$$

which yields:

$$u = 0.64 \text{ approximately} \qquad (2.4.1.36)$$

With these values of B, C, p, the stresses and the velocities become:

$$s = -2h\frac{x}{u^2} \qquad (0 \le x \le 1) \qquad (2.4.1.37)$$

$$n_\phi = -\frac{1}{2} - h\frac{Z}{R}\frac{x^2}{u^2} \qquad (0 \le x \le 1) \qquad (2.4.1.38)$$

$$n_\theta = -\frac{1}{2} + h\frac{Z}{R}\frac{x^2}{u^2} \qquad (0 \le x \le 1) \qquad (2.4.1.39)$$

$$m_\phi = \frac{1}{2}\frac{u^2-x^2}{u^2} \qquad (0 \le x \le u) \qquad (2.4.1.40a)$$

$$m_\phi = \log\frac{x}{u} - \frac{x^2-u^2}{u^2} \qquad (u \le x \le 1) \qquad (2.4.1.40b)$$

$$m_\theta = \frac{1}{2}\frac{u^2+x^2}{u^2} \qquad (0 \le x \le u) \qquad (2.4.1.41a)$$

$$m_\theta = 1 + \log\frac{x}{u} - \frac{x^2-u^2}{u^2} \qquad (0 \le x \le u) \qquad (2.4.1.41b)$$

$$\dot{v} = 0$$

$$\dot{w} = \dot{w}_o\left(1 - \frac{1}{1-2\log u}\frac{x^2}{u^2}\right) \qquad (0 \leq x \leq u) \qquad (2.4.1.42a)$$

$$\dot{w} = -\dot{w}_o\,\frac{2\log x}{1-2\log u} \qquad (u \leq x \leq 1) \qquad (2.4.1.42b)$$

5. Admissibility

It can be verified that the solution obtained is kinematically and statically admissible. Therefore it is an exact solution according to the yield surface adopted.

From the velocity expressions (2.4.1.42a,b), we have: $(\dot{w}')_{x=0} = 0$, and therefore the apex is a regular point, and

$(\dot{w}')_{x=1} = -\dfrac{2\dot{w}_o}{1-2\log u} \neq 0$: there is a hinge at $x = 1$ with $\Delta\dot{w} > 0$ and $\dot{k}_\phi < 0$ as required

2.4.2 Dynamic Response of Simply Supported Shells

1. Solution

If the pressure is not too high, the yield regime will be the same as in the static problem: regime 3-5.

The flow rules and the boundary conditions are the same as in the static problem. Therefore the velocity expressions are the same as in (2.4.1.20) and (2.4.1.21) but now $\dot{w}_o$ is a function of time.

For regime 3-5, relations (2.4.1.1), (2.4.1.2) still apply.

From equation of motion (2.1.2.3), using (2.4.1.1) and having y' and y" from (2.2.1.30) and (2.2.1.31), we have, with (2.4.1.21):

$$(xs)' = \gamma\ddot{w}_o(x-x^3) - (p - \frac{2Z}{R})x$$

Integrating this equation from 0 to x, we have:

$$xs = \gamma\ddot{w}_o(\frac{1}{2}x^2 - \frac{1}{4}x^4) - \frac{1}{2}(p - \frac{2Z}{R})x^2 \quad (2.4.2.1)$$

From equation (2.1.2.1), and using relation (2.4.1.2), we have:

$$h(xm_\phi' + 2m_\phi) = h + xs$$

With xs from (2.4.2.1), we have, after some manipulations:

$$h\frac{d}{dx}(x^2 m_\phi) = hx + \gamma\ddot{w}_o(\frac{1}{2}x^3 - \frac{1}{4}x^5) - \frac{1}{2}(p - \frac{2Z}{R})x^3$$

Integrating this equation from 0 to x, and simplifying by x^2, we obtain:

$$m_\phi = \frac{1}{2} + \frac{\gamma\ddot{w}_o}{h}(\frac{1}{8}x^2 - \frac{1}{24}x^4) - \frac{1}{8h}(p - \frac{2Z}{R})x^2 \quad (2.4.2.2)$$

Using the boundary condition at x = 1 for simply supported shell, we obtain:

$$\gamma\ddot{w}_o = \frac{3}{2}(p - \frac{2Z}{R} - 4h)$$

From the expression of p_s in (2.4.1.8), we have:

$$\frac{2Z}{R} = p_s - 4h \qquad (2.4.2.3)$$

Then:

$$\gamma \ddot{w}_o = \frac{3}{2}(p-p_s) \qquad (2.4.2.4)$$

With $\gamma \ddot{w}_o$ known, we have, from (2.4.2.1) and (2.4.2.3):

$$xs = \frac{p-p_s-8h}{4} x^2 - \frac{3(p-p_s)}{8} x^4 \quad (2.4.2.5)$$

and from (2.4.2.2) and using (2.4.2.3) we have:

$$m_\phi = \frac{1}{2} + [\frac{p-p_s-8h}{16h} x^2 - \frac{p-p_s}{16h} x^4] \qquad (2.4.2.6)$$

Then, from relation (2.4.1.2), we have:

$$m_\theta = \frac{1}{2} - \frac{p-p_s-8h}{16h} x^2 - \frac{p-p_s}{16h} x^4] \qquad (2.4.2.7)$$

From equilibrium equation (2.1.2.2), with $\gamma \ddot{v}$ from (2.4.1.20), and using relation (2.4.1.1) and with y'' from (2.2.1.31) and xs from (2.4.2.5) we have, after some manipulations:

$$\frac{d}{dx}(x^2 n_\phi) = -x + \frac{2Z}{R} [\frac{p-p_s-8h}{4} x^3 - \frac{3(p-p_s)}{8} x^5]$$

Integrating this equation from 0 to x and simplifying by x^2, we obtain:

$$n_\phi = -\frac{1}{2} + \frac{2Z}{R} [\frac{p-p_s-8h}{16} x^2 - \frac{(p-p_s)}{16} x^4] \qquad (2.4.2.8)$$

Then, from (2.4.1.1), we have:

$$n_\theta = -1 - \frac{2Z}{R}\left[\frac{p-ps-8h}{16}x^2 - \frac{p-p_s}{16}x^4\right] \qquad (2.4.2.9)$$

Integrating (2.4.2.4) with respect to τ, and using the initial condition: at $\tau = 0$, $\gamma\dot{w}_o = 0$, we have:

$$\gamma\dot{w}_o = \frac{3}{2}[p_o(1-e^{-\tau}) - p_s\tau] \qquad (2.4.2.10)$$

With this result, the velocities become:

$$\gamma\dot{v} = 0$$

$$\gamma\dot{w} = \frac{3}{2}[p_o(1-e^{-\tau}) - p_s\tau](1-x^2) \qquad (2.4.2.11)$$

2. Admissibility

a) Kinematic Admissibility

From the static problem, we have seen that the solution is kinematically admissible for:

$$\gamma\dot{w}_o \geq 0$$

or, from (2.4.2.10):

$$p_o(1-e^{-\tau}) - p_s\tau \geq 0$$

$$\text{or } \tau \leq \tau_f$$

At $\tau = \tau_f$ determined by:

$$p_o(1-e^{-\tau_f}) - p_s\tau_f = 0\ , \qquad (2.4.2.12)$$

$$\gamma\dot{w}_o = 0,\ \text{then}\ \gamma\dot{w} = 0,$$

and the motion ceases.

b) Static Admissibility

From the expressions (2.4.2.6)-(2.4.2.9), we can see that we can write the stresses in the following forms:

$$m_\phi = 1/2 + P(x) \tag{2.4.2.13}$$

$$m_\theta = 1/2 - P(x) \tag{2.4.2.14}$$

$$n_\phi = -1/2 + \frac{2Z}{R} h\, P(x) \tag{2.4.2.15}$$

$$n_\theta = -1/2 - \frac{2Z}{R} h\, P(x) \tag{2.4.2.16}$$

where

$$P(x) = \frac{p-p_s-8h}{16h} x^2 - \frac{p-p_s}{16h} x^4 \tag{2.4.2.17}$$

The conditions for static admissibility of regime 3-5 are:

$$0 \leq m_\phi \leq 1 \tag{2.4.2.18}$$

$$0 \leq m_\theta \leq 1 \tag{2.4.2.19}$$

$$-1 \leq n_\phi \leq 0 \tag{2.4.2.20}$$

$$-1 \leq n_\theta \leq 0 \tag{2.4.2.21}$$

From the expressions of m_ϕ, m_θ, n_ϕ, n_θ in (2.4.2.13)-(2.4.2.16) and observing that $(2Z/R)h$ is small compared to 1, we see that conditions (2.4.2.18)-(2.4.2.21) can all be satisfied if we have:

$$-\frac{1}{2} \leq P(x) \leq \frac{1}{2} \tag{2.4.2.22}$$

From the expression of P(x) in (2.4.2.17), we have:

$$P'(x) = \frac{x}{8h}[(p-p_s-8h) - 2(p-p_s)x^2] \qquad (2.4.2.23)$$

$P'(x) = 0$ for $x_1 = 0$, and

$$x_2^2 = \frac{p-p_s-8h}{2(p-p_s)} \qquad (2.4.2.24)$$

i. If $p_o - p_s \leq 8h$, then $p-p_s-8h \leq 0$ since p decreases with τ. We have 2 cases:

For $p = p_o e^{-\tau} \geq p_s$ or $\tau \leq \log p_o/p_s$, the right-hand side of (2.4.2.24) is negative and x_2 is imaginary. From (2.4.2.23) we can see that P'(x) is negative: P(x) decreases monotonically from P(0) = 0 to P(1) = -1/2 and (2.4.2.22) is satisfied.

For $p = p_o e^{-\tau} \leq p_s$ or $\log p_o/p_s \leq \tau \leq \tau_f$ where τ_f is the instant when the motion ceases, and is defined in (2.4.2.12), the right-hand side of (2.4.2.24) is positive and x_2 exists:

$$x_2 = \left[\frac{p-p_s-8h}{2(p-p_s)}\right]^{1/2}$$

and we have the following tables of variation of P(x) depending on the position of x_2 with respect to 1.

x	0		x_2		1
P'(x)		−		+	
P(x)	0	↘		↗	−1/2

x	0		1		x_2
P'(x)		−		−	
P(x)	0	↘	−1/2	↘	

From these tables, we can see that if $x_2 < 1$, $P(x)$ will have a minimum at x_2 which is smaller than $-1/2$ and is not admissible. If $x_2 \geq 1$, then $P'(x)$ is negative in the interval $(0,1)$ and $P(x)$ decreases monotonically from $P(0) = 0$ to $P(1) = -1/2$, (2.4.2.22) is satisfied. Thus, to have (2.4.2.22) satisfied, we must have:

$$x_2 \geq 1$$

or, from (2.4.2.24)

$$p - p_s + 8h \geq 0$$

Since p decreases monotonically, this condition is satisfied at $0 \leq \tau \leq \tau_f$ if it is at $\tau = \tau_f$, and we have the condition:

$$p_o e^{-\tau_f} - p_s + 8h \geq 0 \qquad (2.4.2.25)$$

$$(p_o - p_s \leq 8h)$$

A numerical approach has been used for $p_o - p_s$ varying from 0.4h to 8.0h and for shells whose depth varies from 1 to 5 times their thickness. It has been found that (2.4.2.25) is satisfied.

ii) If $p_o - p_s > 8h$, then we have 3 cases:

(α) $p_o e^{-\tau} - p_s \geq 8h$ or $0 \leq \tau \leq \log\,[p_o/(p_s+8h)]$

The right-hand side of (2.4.2.24) is positive and x_2 exists and is smaller than 1.

From the expression of $P'(x)$ in (2.4.2.23), we have the following table of variation of $P(x)$

x	0		x_2		1
$P'(x)$		$+$		$-$	
$P(x)$	0	↗		↘	$-1/2$

Thus $P(x)$ has a maximum at x_2, and from (2.4.2.17), this maximum is:

$$P(x_2) = \frac{x_2^{\,2}}{16h}(p-p_s-8h) - (p-p_s)x_2^2]$$

or with x_2 from (2.4.2.24):

$$P(x_2) = \frac{(p-p_s-8h)^2}{64h(p-p_s)}$$

For (2.4.2.22) to be satisfied, we must have:

$$\frac{(p-p_s-8h)^2}{64h(p-p_s)} \leq \frac{1}{2}$$

or, since $p-p_s > 0$:

$$(p-p_s)^2 - 16h(p-p_s) + 64h^2 \leq 32h(p-p_s)$$

This condition will be satisfied if we have, with $8h < p-p_s$:

$$p-p_s \leq 16(1.5 + \sqrt{2})\ h$$

Since p decreases with τ, this inequality is satisfied for $\tau > 0$ if it is at $\tau = 0$. Thus, we must have:

$$p_o - p_s \leq 16(1.5 + \sqrt{2})\ h \qquad (2.4.2.26)$$

(β) $0 \leq p - p_s \leq 8h$ or: $\log p_o/p_s + 8h \leq \tau \leq \log p_o/p_s$.

For this interval, we have:

$$p - p_s > 0 \text{ and } p - p_s - 8h < 0$$

From (2.4.2.24), we can see that its right-hand side is negative; x_2 is imaginary. From (2.4.2.23), we see that $P'(x)$ is then negative and $P(x)$ decreases monotonically from $P(0) = 0$ to $P(1) = -1/2$, and condition (2.4.2.22) is satisfied.

(γ) $p - p_s < 0$ or $\log p_o/p_s \leq \tau \leq \tau_f$.

In this case, the condition of admissibility is the same as that of the second part of i), which is condition (2.4.2.25).

iii) Conclusion.

From the previous study, we conclude that the solution obtained is admissible for the lesser of the 2 values of p_o determined by (2.4.2.25) and (2.4.2.26) which are:

$$p_o e^{-\tau_f} - p_s + 8h \geq 0 \qquad (2.4.2.25)$$

$$p_o - p_s \leq 16(1.5 + \sqrt{2})h \qquad (2.4.2.26)$$

Condition (2.4.2.25) has been found to be much more restrictive than (2.4.2.26).

The maximum value p_{oL} of p_o for condition (2.4.2.25) can be obtained by solving the simultaneous system of equations made of (2.4.2.12) which is:

$$p_{oL}(1-e^{-\tau_f}) - p_s\tau_f = 0 \qquad (2.4.2.12)$$

and $$p_{oL}e^{-\tau_f} - p_s + 8h = 0 \qquad (2.4.2.25a)$$

the unknown being τ_f and p_{oL}.

3. Final Displacement

Integrating (2.4.2.11) from 0 to τ_f, with the initial condition: $\gamma w = 0$ at $\tau = 0$, we have:

$$\gamma w_f = \frac{3}{2}[(p_o-p_s)\tau_f - \frac{1}{2}p_s\tau_f^2](1-x^2) \qquad (2.4.2.27)$$

The displacement distribution is parabolic and there are no singularities at $x = 0$ (no hinge). The displacement is maximum at the central point $x = 0$ and its value is:

$$\gamma w_o = \frac{3}{2}[(p_o-p_s)\tau_f - \frac{1}{2}p_s\tau_f^2]$$

4. Energy Absorbed

As in section 2.2.2.1, the energy absorbed is given by:

$$E_{abs} = 2\pi R^2 N_o \int_o^{\tau_f} \int_o^1 p\dot{w}\, x dx\, d\tau$$

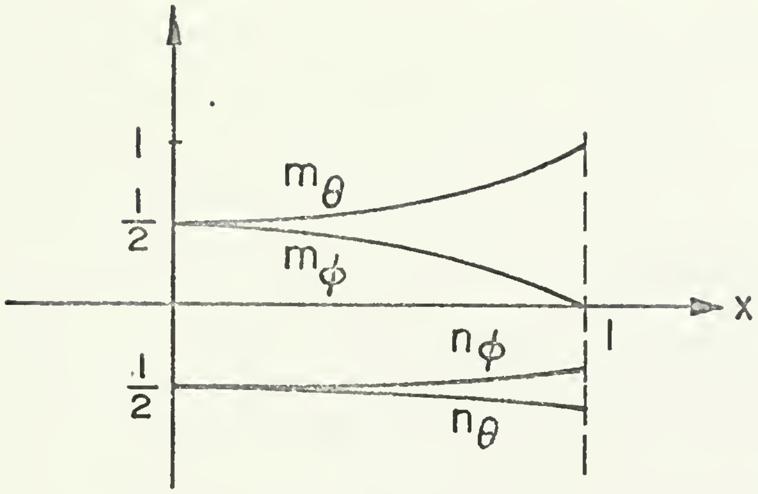
With $p = p_o e^{-\tau}$ and $\dot{w}$ from (2.4.2.11), we obtain:

$$E_{abs} = \frac{\pi R^2 N_o}{2\gamma} \left\{ \frac{p_o^2}{2}(1-e^{-\tau_f})^2 - p_o p_s [1-(1+\tau_f)e^{-\tau_f}] \right\}$$

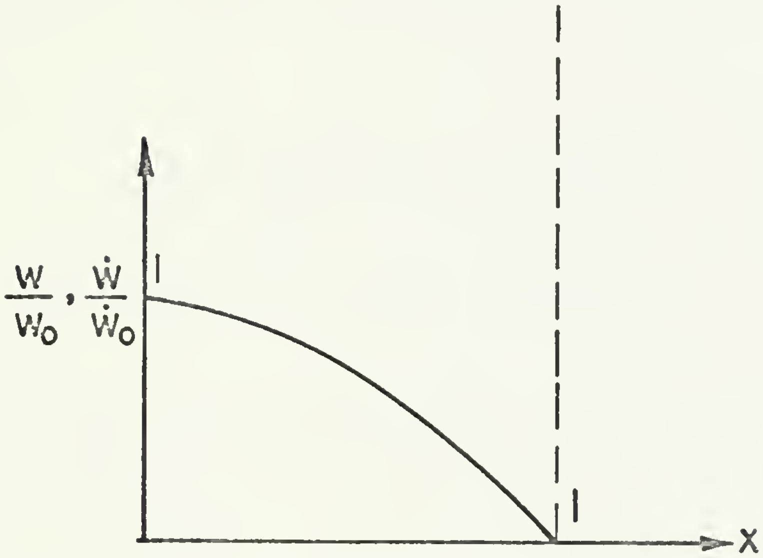
Comparing this expression with (2.2.2.1.28) in the low pressure case of the two-moment limited interaction yield surface, we see that for the same values of p_o and p_s, we have:

$$(E_{abs})_{\substack{\text{uncoupled}\\ \text{square}}} = \frac{3}{4}(E_{abs})_{\substack{\text{two-moment}\\ \text{limited interaction}}}$$

The same conclusion applies to the maximum central deflection.



(a) Stress Distribution



(b) Velocity and Displacement Distribution

Stress and Velocity, Displacement Distributions

Shallow Shells - Uncoupled Diamond Yield Surface

Fig. 20

2.5 Comparison of the Results

2.5.1 Two-moment Limited and Uncoupled Square Yield Surfaces

The static collapse pressures are the same for both yield surfaces, in the case of simply supported shells. The solutions for the dynamic problem can be applied to both yield surfaces, with the only difference that the upper limit p_{oL} of the peak value p_o of the applied pressure is higher for the uncoupled square yield surface. In the case of the two-moment limited interaction yield surface, this limit p_{oL} is solution of a simultaneous system of equations made of (2.2.2.1.8) and (2.2.2.1.19) which are, respectively:

$$p_{oL}(1-e^{-\tau_f}) - p_s\, f = 0$$

$$p_{oL}e^{-\tau_f} - p_s + 12h = 0$$

In the case of the uncoupled square yield surface, this limit p_{oL} is solution of a simultaneous system of equations made of (2.2.2.1.8) and (2.2.2.1.26) which are, respectively:

$$p_{oL}(1-e^{-\tau_f}) - p_s\, f = 0$$

$$p_{oL}e^{-\tau_f} - p_s + 18h = 0$$

Graph G.9 presents these values of p_{oL} as function of the ratio depth/thickness.

2.5.2 Two-moment Limited and Uncoupled Diamond Yield Surfaces

For the uncoupled diamond yield surface, it has been found that the maximum value of the pressure difference p_o-p_s under which the whole shell collapses in one regime is much larger compared to that of the two-moment limited interaction yield surface. In fact, for some values of this pressure difference, we are in the medium high pressure range of the two-moment limited interaction yield surface.

The meridional component of the velocity is zero according to the uncoupled square yield surface whereas it is of the order $0(Z/R)$ compared to the normal component according to the two-moment limited interaction yield surface and is not negligible if we want to be consistent with the shallow shell approximations adopted.

Another aspect of the problem is the expressions of the stress resultants. In the case of the uncoupled diamond yield surface, all the stress and moment resultants are expressible in terms of a function $P(x)$ defined in (2.4.2.17) and the problem of static admissibility can be carried out by studying only this function $P(x)$. This fact simplifies the whole problem considerably, for we might have noted that the problem of determining the admissibility of the solution is also as involved as the problem of obtaining a solution for the whole problem.

One disadvantage of the uncoupled diamond yield surface, however, is that it is not as accurate as the two-moment limited interaction yield surface as suggested by inspection of the bounds in (2.1.3.1) and (2.1.3.3). Since we have here two yield surfaces, one is more accurate than the other, We have also used a method proposed by P. G. Hodge and B. Paul [13] to find a better approximation for the linearized yield surfaces: a peak value of the applied dynamic pressure is chosen, which is 26h in this case. Then the graphs of the maximum central deflection and of the energy absorbed according to the two yield surfaces are concurrently drawn as functions of the depth of the shell which varies from 0 to 1.25 times its thickness. According to the choice of the uncoupled diamond yield surface, which is considered here as the approximate yield surface, with respect to the two-moment limited interaction yield surface, which is considered here as the improved yield surface, we have 3 cases: the approximate yield surface is made to inscribe, circumscribe the improved yield surface or to have the same static collapse pressure as that of the improved yield surface. Each choice will give a curve of variation of the maximum central deflection and one of the energy absorbed. By inspecting the position of these curves relative to the corresponding ones given by the improved yield surface, we can see which alternative gives a curve that is closer to the corresponding one of the

improved yield surface and therefore decide which approximation is better.

From the graphs G. 7,8 we found as in Ref. [13] that the approximation that makes the static collapse pressure equal to that of the improved yield surface gives the best result (except for the energy absorbed in the case of very shallow shell) and that the inscribed and circumscribed approximate yield surface give the upper and lower bounds respectively for the maximum central deflection and energy absorbed.

Because of the simplicity of the solution obtained with the uncoupled diamond yield surface for shells whose reference surface is a quadric, it is natural that we should try to extend the value of the upper limit p_{oL} of the peak value of the applied pressure by allowing the shell to collapse under more than one regime. This attempt has been carried out: at a pressure p_o larger than p_{oL} determined by the system made of (2.4.2.12) and (2.4.2.25a), it was assumed that the shell would collapse in one regime 3-5 (Fig. 6a,b) for $0 \leq \tau \leq \tau_1$ where τ_1 is determined by

$$p_o e^{-\tau_1} - p_s + 8h = 0$$

(At $\tau = \tau_1$, the moment m_ϕ will reach a minimum equal to zero at $x = 1$.)

For $\tau > \tau_1$, m_ϕ would become negative for $u \leq x \leq 1$ and the shell would collapse in more than one regime. We first

assume the following collapse regimes (Fig. 6a,b):

i) $0 \leq x \leq u$: regime 3-5

ii) $u \leq x \leq 1$: regime 3-8

The function $u(\tau)$ has been found to be solution of an ordinary differential equation of second order.

The initial values have been found to be:

at $\tau = \tau_1$, $u(\tau_1) = 1$ which is expected, and:

$$\dot{u}(\tau_1) = \frac{8h}{3[p_o(1-e^{-\tau_1}) - p_s\tau_1]} > 0$$

which is not possible since u should shrink from its initial value $u(\tau_1) = 1$ to a smaller value for $\tau > \tau_1$ and this will require $u(\tau_1)$ to be negative.

Then, another combination has been tried:

i) $0 \leq x \leq u_1$: regime 3-5

ii) $u_1 \leq x \leq u_2$: regime 3-E

iii) $u_2 \leq x \leq 1$: regime 3-8

but did not give any meaningful result.

3. GENERAL SPHERICAL SHELLS

3.1 General Relations

3.1.1 Strain Rate-Velocity Relations

For spherical shells loaded symmetrically, the relations between the generalized strain rates and velocities are [17]:

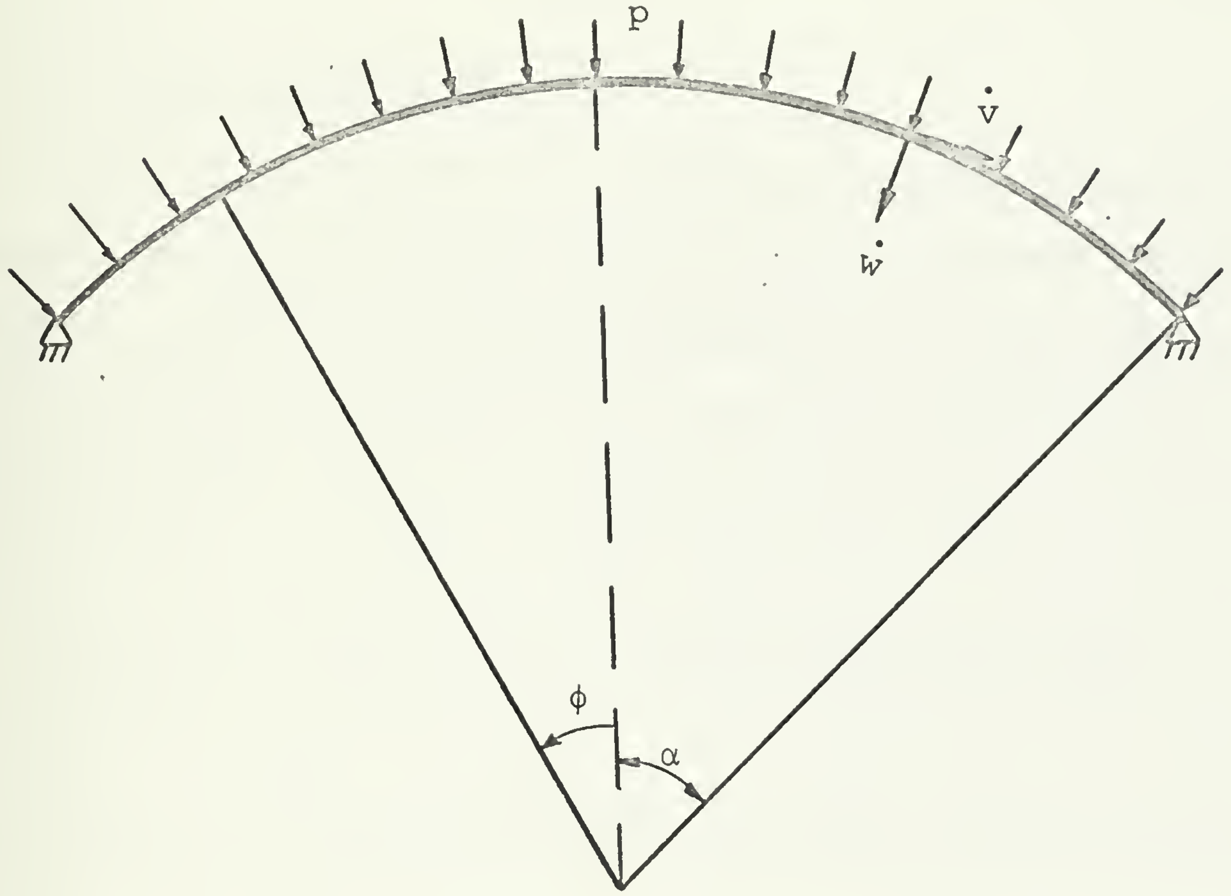


Fig. 18

$$\dot{e}_\theta = \dot{v} \cot \phi - \dot{w} \tag{3.1.1.1}$$

$$\dot{k}_\theta = -h \cot \phi(\dot{v} + \dot{w}') \tag{3.1.1.2}$$

$$\dot{e}_\phi = \dot{v}' - \dot{w} \tag{3.1.1.3}$$

$$\dot{k}_\phi = -h(\dot{v} + \dot{w}') \tag{3.1.1.4}$$

with $(\ )' = d(\ )/d\phi$

and $$h = M_o/AN_o \tag{3.1.1.5}$$

In this case, the reference length A is the radius R of the shell:

$$A = R$$

Then from (2.1.1.5):

$$h = \frac{M_o}{R\,N_o}$$

$$h = \frac{H}{2R} \qquad (3.1.1.6)$$

3.1.2 Equilibrium Equations

Using the results from [17] for spherical shells loaded symmetrically, with some modifications to include the inertia terms, we have the following expressions for the rates of internal energy dissipation $\dot{d}_{int}$ and of external energy input $\dot{d}_{ext}$:

$$\dot{d}_{int} = \int_o^{\alpha} (n_\theta \dot{e}_\theta + n_\phi \dot{e}_\phi + m_\theta \dot{k}_\theta + m_\phi \dot{k}_\phi) \sin\phi \, d\phi$$

$$d_{ext} = \int_o^{\alpha} [(p_n - \gamma \ddot{w})\dot{w} \sin\phi + (p_\phi - \gamma \ddot{v})\dot{v} \sin\phi] \sin\phi \, d\phi$$

$$+ \left\{ \sin\phi [\bar{n}_\phi \dot{v} + \bar{s}\dot{w} - h\bar{m}_\phi (\dot{v} + \dot{w}')] \right\}_{\phi=\alpha}$$

where p_n and p_ϕ are respectively the normal and meridional components of the applied load.

With $\dot{e}_\theta$, $\dot{e}_\phi$, $\dot{k}_\theta$, $\dot{k}_\phi$ from (3.1.1.1)-(3.1.1.4) and applying the principle of virtual velocities, we obtain the

following equations of equilibrium:

$$s \sin\phi = h[(m_\phi \sin\phi)' - m_\theta \cos\phi] - 2hp_\phi \sin\phi + \frac{4}{3}\gamma h^2 \ddot{\Omega}_\phi \sin\phi$$

$$(n_\phi \sin\phi)' - n_\theta \cos\phi - s\sin\phi + p_\phi \sin\phi = \gamma\ddot{v}\sin\phi$$

$$(n_\theta + n_\phi)\sin\phi + (s\sin\phi)' + p_n \sin\phi = \gamma\ddot{w}\sin\phi$$

$\dot{\Omega}_\phi = \dot{v} + \dot{w}'$ is the rate of slope change or angular velocity

As in the case of shallow shells, we observe that $\dot{\Omega}_\phi$ has a sense of rotation contrary to that of positive m_ϕ and the term $\frac{4}{3}\gamma h^2 \dot{\Omega}_\phi = -i(-\ddot{\Omega}_\phi)$, where i is the non-dimensional moment of inertia of unit area of the shell with respect to the reference surface, represents the rotary inertia.

The term $-2hp_\phi$ represents a momentarising from the fact that the dimensional meridional component P_ϕ of the applied load P is acting at a distance H from the reference surface.

If the load is applied only in the normal direction, which is our case, we have $p_\phi = 0$, $p = p_n$, and if the rotary inertia is neglected, then we have

$$s\sin\phi = h[(m_\phi \sin\phi)' - m_\theta \cos\phi] \quad (3.1.2.1)$$

$$(n_\phi \sin\phi)' - n_\theta \cos\phi - s\sin\phi = \gamma\ddot{v}\sin\phi \quad (3.1.2.2)$$

$$(n_\theta + n_\phi)\sin\phi + (s\sin\phi)' + p\sin\phi = \gamma\ddot{w}\sin\phi \quad (3.1.2.3)$$

For a shell without a hole, simply supported at the edge $\phi = \alpha$ and made of isotropic material, we have the following boundary conditions:

At $\phi = 0$:

From isotropy considerations:

$$n_\theta = n_\phi \quad , \quad m_\theta = m_\phi$$

From symmetry considerations:

$$s = 0$$

$\dot{v} = 0, \quad \dot{w}' = 0$ or there will be a hinge

At $\phi = \alpha$:

For simply supported edge, not free to move:

$$m_\phi = 0$$

$\dot{w} = 0, \quad \dot{v} = 0$ or there will be a hinge

The initial conditions are:

At $\tau = 0$, $\dot{v} = v = 0$, $\dot{w} = w = 0$

3.2 Dynamic Plastic Response of Simply Supported Spherical Shells Made of Rigid-Perfectly Plastic Material that Obeys the Two-Moment Limited-Interaction Yield Surface

As in the case of shallow shells, the applied dynamic pressure decays exponentially with time. Its analytic representation in terms of non-dimensional quantities is:

$$p = p_o e^{-\tau}$$

where the peak pressure p_o is larger than the static collapse pressure.

3.2.1. Static Collapse Pressures

Hodge has obtained an exact solution according to the two-moment limited-interaction yield surface for simply supported and clamped spherical caps [18].

The static collapse pressures are found to be:

For simply supported spherical cap:

The yield regime is 5-7 (Figs. 4a,b) and

$$p_s = 2 + \frac{2h \sin \alpha}{\log \dfrac{1 + \sin \alpha}{\cos \alpha} - \sin \alpha} \qquad (3.2.1.1)$$

For clamped spherical cap:

The yield regimes are: $0 \leq \phi \leq \phi_1$: 5-7, $\phi_1 \leq \phi \leq \alpha$: 5-β (Figs. 4a,b)

$$p_{sc} = 2 + \frac{2h \sin \phi_1}{\log \dfrac{1 + \sin \phi_1}{\cos \phi_1} - \sin \phi_1} \qquad (3.2.1.2)$$

where ϕ_1 is solution of:

$$\frac{\sin \phi_1}{\log \dfrac{1 + \sin \phi_1}{\cos \phi_1} - \sin \phi_1} = \frac{1 + \log (\sin \alpha / \sin \phi_1)}{\log (\cos \phi_1 \, / \, \cos \alpha)} \tag{3.2.1.3}$$

These results are valid for spherical caps whose aperture angle α satisfies the condition:

$$\frac{\cos^2 \alpha}{\sin^3 \alpha} \left[\log \frac{1 + \sin \alpha}{\cos \alpha} - \sin \alpha\right] \geq h \tag{3.2.1.4}$$

which is required so that the value $n_\theta(\alpha)$ of n_θ at $\phi = \alpha$ remains negative as required by the yield before profile selected.

3.2.2 Dynamic Response of Simply Supported Spherical Shells

Sankaranarayanan has obtained an exact solution according to the two-moment limited-interaction yield surface for a simply supported spherical cap under dynamic loading [14]. The yield regime is 5-7 (Figs. 4a,b) and the results are:

$$n_\phi = -1, \quad m_\theta = 1$$

$$s = \frac{2-p}{2} \tan \phi + \gamma \ddot{w}_o \left\{ \frac{1 - \cos \phi}{\sin\phi \cos \phi} + \frac{1}{\log (\sec \alpha + \tan \alpha)} \times \left[\frac{\log(\sec \phi + \tan \phi)}{\sin \phi} - \frac{\phi}{\sin \phi \cos \phi}\right] \right\} \tag{3.2.2.1}$$

$$m_\phi = -\frac{1}{h}\left\{(1-h) + \frac{(p-2)\log(\sec\phi+\tan\phi)}{2\sin\phi} - \frac{p}{2}\right\}$$

$$+\frac{\gamma\ddot{w}_o}{h\sin\phi}\left\{\log(\sec\phi+\tan\phi) - \frac{2\int_0^\phi x\sec x\,dx}{\log(\sec\alpha+\tan\alpha)} - \phi\left[1-\frac{\log(\sec\phi+\tan\phi)}{\log(\sec\alpha+\tan\alpha)}\right]\right\}$$

(3.2.2.2)

$$n_\theta = -1 + \frac{p-2}{2}\tan^2\phi + \frac{\gamma\ddot{w}_o}{\cos^2\phi}\left\{\cos^3\phi\left[1-\frac{\log(\sec\phi+\tan\phi)}{\log(\sec\alpha+\tan\alpha)}\right]\right.$$

$$\left. - \left[1-\frac{\phi}{\log(\sec\alpha+\tan\alpha)}\right]\right\} \qquad (3.2.2.3)$$

$$\gamma\ddot{w}_o = \frac{[\log(\sec\alpha+\tan\alpha)][\log(\sec\alpha+\tan\alpha)-\sin\alpha](p-p_s)}{2\left\{[\log(\sec\alpha+\tan\alpha)]^2 - 2\int_0^\alpha x\sec x\,dx\right\}}$$

(3.2.2.4)

$$\gamma\dot{w} = \gamma\dot{w}_o\cos\phi\left[1-\frac{\log(\sec\phi+\tan\phi)}{\log(\sec\alpha+\tan\alpha)}\right] \qquad (3.2.2.5)$$

$$\gamma\dot{v} = \gamma\dot{w}_o\sin\phi\left[1-\frac{\log(\sec\phi+\tan\phi)}{\log(\sec\alpha+\tan\alpha)}\right] \qquad (3.2.2.6)$$

$\gamma\dot{w}_o$ is the velocity of the apex and is given by:

$$\gamma\dot{w}_o = \frac{[\log(\sec\alpha+\tan\alpha)]\,[\log(\sec\alpha+\tan\alpha)-\sin\alpha]\,[p_o(1-e^{-\tau})-p_s\tau]}{2\left\{[\log(\sec\alpha+\tan\alpha)]^2 - 2\int_0^\alpha x\sec x\,dx\right\}}$$

(3.2.2.7)

The solution is kinematically admissible for $0 \le \tau \le \tau_f$ where τ_f is the instant when the motion ceases and is

determined by:

$$p_o(1-e^{-\tau_f}) - p_s\tau_f = 0 \qquad (3.2.2.8)$$

The peak value p_o of the dynamic load is such that:

$$p_o \leq 2 + \frac{6h \sin \alpha \log (\sec \alpha + \tan \alpha)}{\log(\sec\alpha+\tan\alpha)[\log(\sec\alpha+\tan\alpha)-3\sin\alpha] + 4\int_0^\alpha x\sec x dx}$$

so that $n_\theta = -1$ be minimum at $\phi = 0$

(3.2.2.9)

Then, it can be verified numerically that the solution is statically admissible.

The final displacement component distributions are given by:

$$\gamma w_f = \gamma w_o \cos \phi \left[1 - \frac{\log(\sec \phi + \tan \phi)}{\log(\sec \alpha + \tan \alpha)}\right] \qquad (3.2.2.10)$$

$$\gamma v_f = \gamma w_o \sin \phi \left[1 - \frac{\log(\sec \phi + \tan \phi)}{\log(\sec \alpha + \tan \alpha)}\right] \qquad (3.2.2.11)$$

Where γw_o is the final central displacement:

$$\gamma w_o = \frac{\log(\sec\alpha+\tan\alpha)[\log(\sec\alpha+\tan\alpha)-\sin\alpha][(p_o-p_s)\tau_f - \frac{1}{2}p_s\tau_f^2]}{2\left\{[\log(\sec\alpha+\tan\alpha)]^2 - 2\int_0^\alpha x \sec x dx\right\}}$$

(3.2.2.12)

where τ_f is given by (3.2.2.8).

The energy absorbed is:

$$E_{abs} = \int_0^{t_f} \int_0^\alpha p\frac{dU_n}{dt} 2\pi R^2 \sin \phi \, d\phi dt \qquad (3.2.2.13)$$

With the non-dimensional quantities:

$$\tau = \frac{t}{T_o}, \quad w = \frac{U_n}{R}, \quad p = \frac{RP}{N_o}$$

we have:

$$E_{abs} = 2\pi R^2 N_o \int_0^{\tau_f} \int_0^{\alpha} p\dot{w} \sin\phi \, d\phi d\tau$$

With (3.2.2.5) and (3.2.2.7), it can be shown that we have:

$$E_{abs} = 2\pi R^2 N_o \frac{[\log(\sec\alpha+\tan\alpha)]\ [\log(\sec\alpha+\tan\alpha)-\sin\alpha]}{2\gamma\left\{[\log(\sec\alpha+\tan\alpha)]^2 - 2\int_0^{\alpha} x\sec x dx\right\}} \times$$

$$\left\{\frac{1}{2} p_o^2 (1-e^{-\tau_f})^2 - p_o p_s [1 - (1+\tau_f)e^{-\tau_f}]\right\} \quad (3.2.2.14)$$

3.2.3 Limited Series Expansion of the Solution of the Dynamic Problem

To have a better idea on the range of validity of the dynamic solution and on the static admissibility, we have made a limited series expansion of the solution obtained in (3.2.2.1)-(3.2.2.7), by neglecting terms of 4th order or higher.

1. Solution

With:

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$$

$$\log (\sec x + \tan x) = x + \frac{x^3}{6} + \frac{x^5}{24} + \frac{61x^7}{5040}$$

$$\log (\sec x + \tan x) - \sin x = \frac{x^3}{3} + \frac{x^5}{30} + \frac{67x^7}{5040}$$

$$\int_0^x \log (\sec u + \tan u)\, du = \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{144}$$

$$\int_0^x u \sec u du = x + \frac{x^3}{18} + \frac{7x^5}{1800} + \frac{31x^7}{105840}$$

and using the results from 3.2.1.1, it can be shown that we obtain:

$$p_s = 2 + \frac{6h}{\alpha^2} (1 - \frac{4\alpha^2}{15}) \qquad (3.2.3.1)$$

The presence of α^2 in the denominator of the second term comes from the fact that in this part, we have chosen the shell radius as the reference length A in the non-dimensionalizing procedure. If the base radius B of the shell is chosen as the reference length instead, then from: $B = R \sin \alpha$, we have:

$$p_s' = p_s \sin \alpha$$

$$h' = \frac{h}{\sin \alpha}$$

Then for small value of α, where $\alpha^2 << 1$, it can be shown that:

$$p_s' = 4 \frac{Z}{B} + 6h$$

as has been found for shallow shells in (2.2.1.28).

The limited series expansion of the dynamic solution is:

$$m_\phi = \frac{\alpha^2-\phi^2}{\alpha^2} - \frac{4(\phi^2-\alpha^2)\phi^2}{15\alpha^2} - \frac{(p-p_s)}{6h}\,\frac{\phi^2(\phi-\alpha)}{\alpha}\left(1 + \frac{15\phi^2-2\alpha\phi-14\alpha^2}{30}\right) \tag{3.2.3.2}$$

$$\begin{aligned} n_\theta = &- 1 + 3h\,\frac{\phi^2}{\alpha^2}\left(1 - \frac{4\alpha^2}{15} + \frac{2\phi^2}{3}\right) \\ &+ \frac{p-p_s}{2}\,\phi^2\left[- 5 + \frac{7\alpha^2}{5} - \frac{7\phi^2}{12} + \frac{\phi}{\alpha}\left(\frac{16}{3} - \frac{32\alpha^2}{15} + \frac{8\phi^2}{3}\right)\right] \end{aligned} \tag{3.2.3.3}$$

$$\gamma\dot{w} = \gamma\dot{w}_o\left[1 - \frac{\phi^2}{2} - \frac{\phi}{\alpha}\left(1 - \frac{\alpha^2}{6} - \frac{\phi^2}{3}\right)\right] \tag{3.2.3.4}$$

$$\gamma\dot{v} = \gamma\dot{w}_o\,\phi\left[1 - \frac{\phi^2}{6} - \frac{\phi}{\alpha}\left(1 - \frac{\alpha^2}{6}\right)\right] \tag{3.2.3.5}$$

$$\gamma\dot{w}_o = 2\left(1 - \frac{7\alpha^2}{30}\right)\left[p_o(1-e^{-\tau}) - p_s\tau\right] \tag{3.2.3.6}$$

2. Admissibility

a) Kinematic

The conditions for kinematic admissibility for regime 5-7 (Figs. 4a,b) are $\dot{e}_\phi \leq 0$ and $\dot{k}_\theta \geq 0$.

For $\dot{e}_\phi$, we have from (3.1.1.3) and (3.2.3.4), (3.2.3.5):

$$\dot{e}_\phi = - \dot{w}_o\,\frac{\phi}{\alpha}\left(1 - \frac{\alpha^2}{2} - \frac{\phi^2}{3\alpha^2}\right)$$

From (3.2.3.6) and since α^2 is smaller than 1, we can see that $\dot{w}_o$ is positive for $\tau < \tau_f$, where τ_f is determined by (3.2.2.8). Therefore for $\dot{e}_\phi$ to be negative, we must have:

$$\frac{\phi^2}{3\alpha^2} + \frac{\alpha^2}{2} < 1 \quad , \quad 0 \leq \phi \leq \alpha$$

or, with $\phi = \alpha$:

$$\alpha^2 < \frac{4}{3}$$

which is satisfied since α^2 is smaller than 1.

For $\dot{k}_\theta$, we have from (3.1.1.2) and (3.2.3.4), (3.2.3.5):

$$\dot{k}_\theta = h \cot \phi \frac{\dot{w}_o}{\alpha} (1 + \alpha\phi^3 - \frac{\alpha^2}{6} - \frac{\alpha^2\phi^2}{6})$$

For $\dot{k}_\theta$ to be positive, with $\phi \leq \alpha < 1$, we must have:

$$\frac{\alpha^2}{6} + \frac{\alpha^2\phi^2}{6} - \alpha\phi^3 < 1$$

With $\alpha^4 << 1$, this inequality is always satisfied.

Therefore the solution is kinematically admissible.

b) Static

i) n_θ: the conditions for admissibility of n_θ are:

$$-1 \leq n_\theta \leq 0$$

From (3.2.3.3), we have:

$$n_\theta' = \phi \times PN(\phi) \qquad (3.2.3.8)$$

with:

$$PN(\phi) = \frac{20}{3} \frac{p-p_s}{\alpha} \phi^3 + [\frac{8h}{\alpha^2} - \frac{7(p-p_s)}{6}]\phi^2 + \frac{40-16\alpha^2}{5}(p-p_s)\phi$$

$$+ h \frac{30-8\alpha^2}{5\alpha^2} - (p-p_s) \frac{25-7\alpha^2}{5} \qquad (3.2.3.9)$$

At $\phi = 0$, we have:

$$n'_\theta = 0 \quad \text{and}$$

$$n_\theta = -1$$

Therefore a necessary condition for n_θ to be admissible is that there be a minimum at $\phi = 0$, or

$$(n''_\theta)_{\phi=0} \geq 0$$

Or, from (3.2.3.8), (3.2.3.9):

$$3h \frac{30-8\alpha^2}{15\alpha^2} - (p-p_s) \frac{25-7\alpha^2}{5} \geq 0$$

Solving for $p-p_s$, we obtain approximately, with $\alpha^4 << 1$:

$$p-p_s \leq \frac{1.2h}{\alpha^2} (1 + \frac{\alpha^2}{75})$$

Since p is monotonically decreasing with τ, this condition is satisfied at $\tau > 0$ if it is at $\tau = 0$ or:

$$p_o-p_s \leq \frac{1.2h}{\alpha^2} (1 + \frac{\alpha^2}{75}) \qquad (3.2.3.10)$$

If the radius B is used as reference length and if $\alpha^2 << 1$, then we find again a condition for static admissibility for the low pressure case of shallow shells.

Moreover, for $p-p_s > 0$, it can be proved that for p_o-p_s satisfying (3.2.3.10) the coefficients of ϕ^3, ϕ^2, ϕ and the constant term in $PN(\phi)$ (3.2.3.9) are all positive. Therefore $PN(\phi)$ is positive for $p-p_s > 0$.

For $p-p_s < 0$, it can be verified numerically that for p_o-p_s satisfying (3.2.3.10) and with $\alpha \leq \pi/6$ (we took $\pi/6$,

since we must have $\alpha^4 << 1$), $PN(\phi)$ is still positive.

Therefore n_θ increases monotonically with ϕ, and for n_θ to be admissible, it is necessary and sufficient that at $\phi = \alpha$, we have:

$$n_\theta < 0 \quad \text{, or:}$$

$$3h(1 + \frac{2\alpha^2}{5}) + \frac{p-p_s}{2}\alpha^2(\frac{1}{3} + \frac{27\alpha^2}{20}) \leq 1 \qquad (3.2.3.11)$$

With p_o-p_s limited by (3.2.3.10) and with h small for thin shell, condition (3.2.3.11) is always satisfied and therefore n_θ is admissible.

ii) m_ϕ: the conditions for admissibility of m_ϕ are:

$$0 \leq m_\phi \leq 1$$

From the expression of m_ϕ in (3.2.3.2), we have:

$$m'_\phi = -\phi \times PM(\phi) \qquad (3.2.3.12)$$

with:

$$PM(\phi) = \frac{5(p-p_s)}{12h\alpha}\phi^3 + [\frac{16}{15\alpha^2} - \frac{17(p-p_s)}{45h}]\phi^2$$
$$+ \frac{p-p_s}{10h}\,\frac{5-2\alpha^2}{\alpha}\phi + \frac{30-8\alpha^2}{15\alpha^2} - \frac{p-p_s}{6h}\,\frac{30-14\alpha^2}{15}$$
$$(3.2.3.13)$$

At $\phi = 0$, we have:

$$m'_\phi = 0 \quad \text{and} \quad m_\phi = 1$$

Therefore a necessary condition for m_ϕ to be admissible is that there be a maximum at $\phi = 0$ or:

$$(m''_{\phi})_{\phi} \leq 0$$

From (3.2.3.12), we have:

$$(m''_{\phi})_{\phi=o} = \frac{p-p_s}{6h} \frac{30-14\alpha^2}{15} \quad \frac{30-8\alpha^2}{15\alpha^2} \leq 0$$

which yields, approximately, with $\alpha^4 << 1$:

$$p-p_s \leq \frac{6h}{\alpha^2} (1 + \frac{\alpha^2}{5})$$

This condition will be satisfied for $\tau > 0$ if we have:

$$p_o-p_s \leq \frac{6h}{\alpha^2} (1 + \frac{\alpha^2}{5}) \qquad (3.2.3.14)$$

Comparing (3.2.3.14) with (3.2.3.10), we see that (3.2.3.14) is satisfied if (3.2.3.10) is.

Since m_{ϕ} must be positive and since $m_{\phi} = 0$ at $\phi = \alpha$, another necessary condition is that:

$$m'_{\phi}(\alpha) \leq 0$$

which yields:

$$\frac{p-p_s}{6h}(1 - \frac{\alpha^2}{30}) + \frac{30+8\alpha^2}{15\alpha^2} \geq 0$$

Since p decreases with τ, this inequality is satisfied for $\tau \leq \tau_f$ if it is at $\tau = \tau_f$ or:

$$\frac{p_o e^{-\tau_f} - p_s}{6h}(1 - \frac{\alpha^2}{30}) + \frac{30+8\alpha^2}{15\alpha^2} \geq 0 \qquad (3.2.3.15)$$

For p_o satisfying (3.2.3.10), it has been verified numerically that this condition is satisfied. However,

(3.2.3.14) and (3.2.3.15) are only necessary conditions. From the expression of m'_ϕ in (3.2.3.12), we see that $m'_\phi = 0$ for $\phi = 0$ and for

$$PM(\phi) = 0$$

where $PM(\phi)$ is defined in (3.2.3.13). Since $PM(\phi) = 0$ may have 3 roots and if none or 2 of them are in the interval $(0,\alpha)$, m_ϕ may still be positive in this interval. A numerical approach is necessary to find the variations of m_ϕ in the interval $(0,\alpha)$. This has not been carried out, since it has been done in [14] for the general case.

It is of some interest to note that if the applied load is a rectangular pulse, we have:

$$p = p_o \text{ for } 0 \leq \tau \leq 1$$

and $$p = 0 \text{ for } 1 \leq \tau \leq \tau_f$$

and for this load, condition (3.2.3.15) becomes:

$$-\frac{p_s}{6h}\left(1 - \frac{\alpha^2}{30}\right) + \frac{30+8\alpha^2}{15\alpha^2} \geq 0$$

with p_s from (3.2.3.1), this condition becomes, with $\alpha^4 << 1$, approximately:

$$\alpha^2\left(1 - \frac{2\alpha^2}{3}\right) \leq 3h \qquad (3.2.3.16)$$

With $\alpha^2 < 1$, this inequality will be satisfied for $\alpha^2 \leq 3h$, or with $h << 1$, α will be small. Then if B is the base radius of the shell, we have, approximately: $\alpha \simeq B/R$.

Then, we must have:

$$\frac{B}{R} \leq 3 \times \frac{2H}{4R}$$

or:

$$\frac{B}{R} \leq \frac{3}{4} \frac{2H}{B}$$

For thin shell, 2H/B is small, and therefore this inequality can be satisfied only by a very shallow shell.

We can also write this equality in the form:

$$B^2 \leq \frac{3}{4} \times 2H \times R$$

We have:

$$B^2 = Z(2R-Z)$$

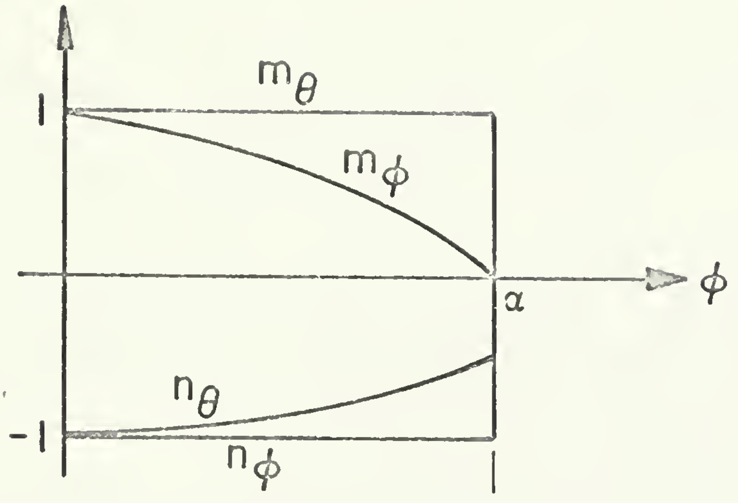
Or, since the shell is shallow,

$$B^2 \simeq 2RZ.$$

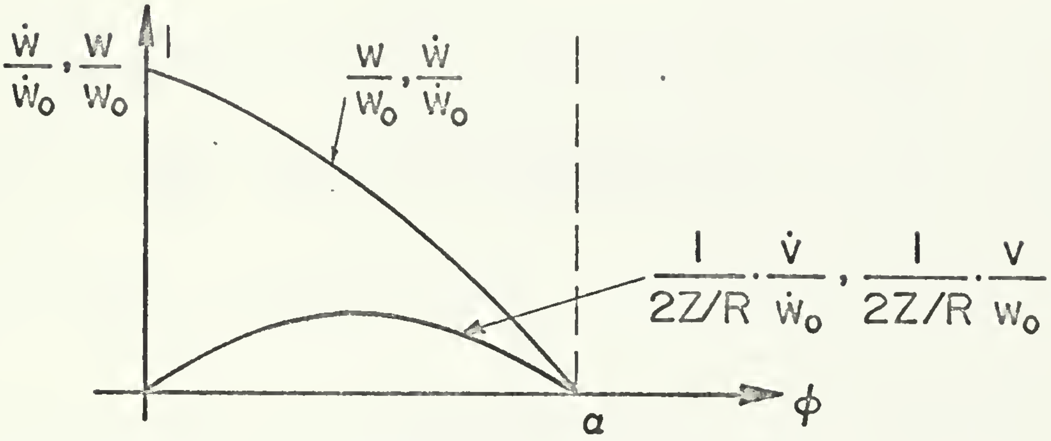
Then condition (3.2.3.16) is equivalent to:

$$Z \leq \frac{3}{8} \times 2H$$

or the height of the shell must be smaller than 3/8 its thickness. This result has previously been found in the case of shallow shell.



(a) Stress Distribution



(b) Velocity and Displacement Distribution

Stress and Velocity, Displacement Distributions

Deep Shell - Two-Moment Limited Interaction Yield Surface

Fig. 21

3.3 Dynamic Plastic Response of Simply Supported Spherical Shells made of Rigid-Perfectly Plastic Material that Obeys the Uncoupled Square Yield Surface

3.3.1 Static Collapse Pressures

The yield regime is 4-5 of Figs. (5a,b) for both simply supported and clamped edge.

In the case of simply supported edge, the collapse pressure may be shown to be the same as (3.2.1) and is given by:

$$p_s = 2 + \frac{2h \sin \alpha}{\log \dfrac{1 + \sin \alpha}{\cos \alpha} - \sin \alpha} \qquad (3.3.1.1)$$

In the case of clamped edge, we still have [18]:

$$m_\phi = 1 - \frac{p-2}{2h} \left[\frac{1}{\sin \phi} \log \frac{1 + \sin \phi}{\cos \phi} - 1\right]$$

At $\phi = \alpha$, $m_\phi = -1$ and we find:

$$p_{sc} = 2 + \frac{4h \sin \alpha}{\log \dfrac{1 + \sin \alpha}{\cos \alpha} - \sin \alpha} \qquad (3.3.1.2)$$

3.3.2 Dynamic Response of Simply Supported Spherical Shells

The yield regime, for the low pressure range, is still 4-5 (Figs. 5a,b). Then we will obtain the same equations and the same boundary and initial conditions as in 3.2.2. Therefore equations (3.2.2.1)-(3.2.2.7) still apply. The range of pressure for which this solution applies is given by (3.2.2.9)

The solution obtained by limited series expansion in (3.2.3.1)-(3.2.3.6) is still valid in the same range of pressure given by (3.2.3.10).

Condition (3.2.3.11) must be replaced by:

$$3h(1 + \frac{2\alpha^2}{5}) + \frac{p-p_s}{2}\alpha^2(\frac{1}{3} + \frac{27\alpha^2}{20}) \leq 2 \qquad (3.3.2.1)$$

which expresses that at $\phi = \alpha$, $n_\theta \leq 1$.

Since m_ϕ can now be negative, condition (3.2.3.15) is now not necessary and is replaced by the one which expresses that the minimum of m_ϕ be larger than -1. This problem must be solved numerically and is not carried out here, since we know that in the range of pressure determined by (3.2.3.10), the solution is an exact one for the more restrictive two-moment limited interaction yield surface.

3.4 Dynamic Plastic Response of Simply Supported Spherical Shells Made of Rigid Perfectly Plastic Material that Obeys the Uncoupled Diamond Yield Surface

3.4.1 Static Collapse Pressures

A. Simply Supported Edge

As in the case of shallow shell, the yield regime is 3-5 of Figs. (6a,b). From table 2.1.3.3, we have the following relations:

$$n_\theta + n_\phi = -1 \tag{3.4.1.1}$$

$$m_\theta + m_\phi = 1 \tag{3.4.1.2}$$

Together with the equilibrium equations (3.1.2.1)-(3.1.2.3), in which the inertia forces are neglected, we have a system of five equations for five unknown generalized stresses.

From equation (3.1.2.3) and relation (3.4.1.1), we have:

$$(s \sin \phi)' = -(p-1) \sin \phi$$

Integrating between 0 and ϕ, we obtain:

$$s \sin \phi = -(p-1)(1 - \cos \phi) \tag{3.4.1.3}$$

From equation (3.1.2.2) and relation (3.4.1.1), we have:

$$n_\phi' \sin \phi + 2n_\phi \cos \phi = -\cos \phi + s \sin \phi$$

Multiplying both sides by $\sin \phi$, we obtain, with $s \sin \phi$ from (3.4.1.3):

$$\frac{d}{d\phi}(n_\phi \sin^2 \phi) = -\sin\phi \cos \phi - (p-1)(\sin \phi - \sin \phi \cos \phi)$$

Integrating between 0 and ϕ and simplifying by $\sin^2\phi$, we obtain:

$$n_\phi = -\frac{1}{2} - (p-1)\left[\frac{1-\cos\phi}{\sin^2\phi} - \frac{1}{2}\right]$$

After some reduction, this equation becomes:

$$n_\phi = -\frac{1}{2} - \frac{p-1}{2}\,\frac{1-\cos\phi}{1+\cos\phi} \qquad (3.4.1.4)$$

Then from relation (3.4.1.1):

$$n_\theta = -\frac{1}{2} + \frac{p-1}{2}\,\frac{1-\cos\phi}{1+\cos\phi} \qquad (3.4.1.5)$$

From (3.1.2.1) and (3.4.1.2), we have:

$$h\,[m_\phi' \sin\phi + 2m_\phi \cos\phi] = h\cos\phi + s\sin\phi$$

Multiplying both sides by $\sin\phi$ and with $s\sin\phi$ from (3.4.1.3), we have:

$$h\,\frac{d}{d\phi}(m_\phi \sin^2\phi) = h\sin\phi\cos\phi - (p-1)(\sin\phi - \sin\phi\cos\phi)$$

Integrating between 0 and ϕ and simplifying by $\sin^2\phi$, we obtain, after some reductions:

$$m_\phi = \frac{1}{2} - \frac{p-1}{2h}\,\frac{1-\cos\phi}{1+\cos\phi} \qquad (3.4.1.6)$$

Then from (3.4.1.2), we have:

$$m_\theta = \frac{1}{2} + \frac{p-1}{2h}\,\frac{1-\cos\phi}{1+\cos\phi} \qquad (3.4.1.7)$$

At $\phi = \alpha$, we have $m_\phi = 0$. Thus, we derive:

$$p = p_s = 1 + h\,\frac{1+\cos\alpha}{1-\cos\alpha} \qquad (3.4.1.8a)$$

$$p_s = 1 + h\cot^2\frac{\alpha}{2} \qquad (3.4.1.8b)$$

If the base radius B is taken as the reference length and if α^2 is neglected with respect to 1, it can be shown that (3.4.1.8a) reduces to the expression of the shallow shell case. (2.4.1.8)

With the expression (3.4.1.8a) for p, equations (3.4.1.4)-(3.4.1.7) become:

$$n_\phi = -\frac{1}{2} - h\, F(\phi) \qquad (3.4.1.9)$$

$$n_\theta = -\frac{1}{2} + h\, F(\phi) \qquad (3.4.1.10)$$

$$m_\phi = \frac{1}{2} - F(\phi) \qquad (3.4.1.11)$$

$$m_\theta = \frac{1}{2} + F(\phi) \qquad (3.4.1.12)$$

where

$$F(\phi) = \frac{1}{2}\,\frac{1+\cos\alpha}{1-\cos\alpha} \times \frac{1-\cos\phi}{1+\cos\phi} \qquad (3.4.1.13a)$$

$$F(\phi) = \frac{1}{2}\,\frac{\tan^2\frac{\phi}{2}}{\tan^2\frac{\alpha}{2}} \qquad (3.4.1.13b)$$

From (3.4.1.9)-(3.4.1.12) and from the fact that h is small compared to 1, we can see that the necessary and sufficient condition for static admissibility is, for yield regime 3-5:

$$-\frac{1}{2} \le F(\phi) \le \frac{1}{2} \quad \text{for} \quad 0 \le \phi \le \alpha$$

From (3.4.1.13b), we can see that $F(\phi)$ increases from 0 to 1/2 as ϕ increases from 0 to α. Therefore the solution is statically admissible.

The flow rules associated with regime 3-5 are, from table 2.1.3.3:

$$\dot{e}_\theta - \dot{e}_\phi = 0 \qquad (3.4.1.14)$$

and $$\dot{k}_\theta - \dot{k}_\phi = 0 \qquad (3.4.1.15)$$

From (3.1.1.1) and (3.1.1.3), equation (3.4.1.14) yields:

$$\dot{v} = A \sin \phi \qquad (3.4.1.16)$$

At $\phi = \alpha$, we have $\dot{v} = 0$. Thus:

$A \sin \alpha = 0$, or since $\sin \alpha \neq 0$:

$$A = 0$$

and $$\dot{v} \equiv 0 \qquad (3.4.1.17)$$

From (3.1.1.2), (3.1.1.4) and with (3.4.1.16), equation (3.4.1.15) yields:

$$\dot{w}' = B \sin \phi$$

Integrating, we obtain:

$$\dot{w} = - B \cos \phi + C \qquad (3.4.1.18)$$

Using the boundary condition at $\phi = \alpha$:

$$\dot{w}(\alpha) = 0,$$

we obtain:

$$\dot{w} = B(\cos \alpha - \cos \phi)$$

Let $\dot{w}_o = -B(1 - \cos \alpha)$, we have then:

$$\dot{w} = \dot{w}_o \frac{\cos \phi - \cos \alpha}{1 - \cos \alpha} \qquad (3.4.1.19)$$

The condition for kinematic admissibility are:

$$\dot{e}_\phi = \dot{e}_\theta \leq 0 \quad \text{and} \quad \dot{k}_\phi = \dot{k}_\theta \geq 0$$

From (3.1.1.1) and with $\dot{v}$ and $\dot{w}$ from (3.4.1.17) and (3.4.1.19), we have:

$$\dot{e}_\phi = \dot{e}_\theta = -\dot{w} = -\dot{w}_o \frac{\cos\phi - \cos\alpha}{1 - \cos\alpha}$$

$$(0 \leq \phi \leq \alpha)$$

which is negative since $\dot{w}_o > 0$ and $\phi \leq \alpha$.

From (3.1.1.2) and with $\dot{v}$ and $\dot{w}$ from (3.4.1.17) and (3.4.1.19), we have:

$$\dot{k}_\phi = \dot{k}_\theta = h\dot{w}_o \frac{\cos\phi}{1 - \cos\alpha}$$

$$(0 \leq \phi \leq \alpha)$$

With $\dot{w}_o > 0$, $\dot{k}_\theta$ will be positive if $\alpha \leq \pi/2$.

Therefore the solution obtained is an exact one for $\alpha \leq \pi/2$ and the static collapse pressure is given by (3.4.1.8a) or (3.4.1.8b).

At $\phi = 0$, $\dot{w}' = 0$ and there is no singularity at the apex of the shell.

B. Clamped Edge

1. Solution

As in the case of shallow shells, we would expect a hinge at the clamped edge, and the yield regimes are (Figs. 6a,b):

$0 \leq \phi \leq \beta$: regime 3-5

$\beta \leq \phi \leq \alpha$: regime 3-8

a) $0 \leq \phi \leq \beta$: regime 3-5

In this interval, the 5 equations to determine the five generalized stresses are still made of (3.4.1.1), (3.4.1.2) and (3.1.2.1)-(3.1.2.3) where the inertia forces are neglected. But we cannot use the boundary condition at $\phi = \alpha$ in this case. Therefore equations (3.4.1.3)-(3.4.1.7) still remain valid.

The flow rules remain the same, and without using the boundary conditions at $\phi = \alpha$, the expressions for the velocities are (3.4.1.16) and (3.4.1.18).

b) $\beta \leq \phi \leq \alpha$: regime 3-8

From table 2.1.3.3, we have the following relations for regime 3-8:

$$n_\theta + n_\phi = -1 \qquad (3.4.1.20)$$

which is the same as (3.4.1.1) and

$$m_\theta - m_\phi = 1 \qquad (3.4.1.21)$$

With (3.4.1.20) which is the same as (3.4.1.1), the equation to determine $s \sin \phi$ is the same as in the simply supported case and since $s \sin \phi$ must be continuous across the boundary $\phi = \beta$, the expression for $s \sin \phi$ is the same as (3.4.1.3)

Similarly, we obtain the same expressions for n_ϕ and n_θ as in (3.4.1.4) and (3.4.1.5).

To determine m_ϕ and m_θ, we have (3.1.2.1) and (3.4.1.21), which give:

$$m_\phi' = \frac{\cos\phi}{\sin\phi} + \frac{1}{h}\frac{s\sin\phi}{\sin\phi}$$

With $s \sin\phi$ from (3.4.1.3), we have:

$$m_\phi' = \frac{\cos\phi}{\sin\phi} - \frac{p-1}{h}\left(\frac{1}{\sin\phi} - \frac{\cos\phi}{\sin\phi}\right)$$

Integrating from β to ϕ, with $m_\phi(\beta) = 0$, since m_ϕ must be continuous at $\phi = \beta$, we obtain:

$$m_\phi = \left(1 + \frac{p-1}{h}\right)\log\frac{\sin\phi}{\sin\beta} - \frac{p-1}{2h}\left(\log\frac{1+\cos\beta}{1-\cos\beta} - \log\frac{1+\cos\phi}{1-\cos\phi}\right) \tag{3.4.1.22}$$

Then:

$$m_\theta = 1 + \left(1+\frac{p-1}{h}\right)\log\frac{\sin\phi}{\sin\beta} - \frac{p-1}{2h}\left(\log\frac{1+\cos\beta}{1-\cos\beta} - \log\frac{1+\cos\phi}{1-\cos\phi}\right) \tag{3.4.1.23}$$

The flow rules for regime 3-8 are, from table 2.1.3.3:

$$\dot{e}_\theta = \dot{e}_\phi \tag{3.4.1.24}$$

$$-\dot{k}_\theta = \dot{k}_\phi \tag{3.4.1.25}$$

From (3.1.1.1) and (3.1.1.3) and with (3.4.1.24), we have:

$$\dot{v} = D\sin\phi$$

Using the boundary condition at $\phi = \alpha$, we have:

$\dot{v} = D\sin\alpha = 0$, and since $\sin\alpha \neq 0$,

we must have:

$$D = 0,$$

thus $\dot{v} = 0 \qquad (\beta \leq \phi \leq \alpha)$

But $\dot{v}$ must be continuous at $\phi = \beta$, and from (3.4.1.16), we have:

$$A \sin \beta = 0$$

Since $\sin \beta \neq 0$, we have:

$$A = 0$$

Thus

$$\dot{v} \equiv 0 \quad (0 \leq \phi \leq \alpha) \tag{3.4.1.26}$$

From (3.1.1.2), (3.1.1.4) and with (3.4.1.25), (3.4.1.26), we have,

$$\dot{w}' = \frac{E}{\sin \phi}$$

Integrating from ϕ to α, with $\dot{w}(\alpha) = 0$, we have:

$$\dot{w} = \frac{E}{2} \left[\log\frac{1 + \cos \alpha}{1 - \cos \alpha} - \log\frac{1 + \cos \phi}{1 - \cos \phi} \right] \tag{3.4.1.27}$$

c) Boundary Matching

At α^-, we have, from (3.4.1.27):

$$\dot{w}' = \frac{E}{\sin \alpha} \neq 0$$

The slope $\dot{w}'$ will be discontinuous at $\phi = \alpha$. This is possible if at $\phi = \alpha$, we have:

$$m_\phi = -1$$

With this condition and from (3.4.1.22), we derive:

$$p = p_s = 1 + \frac{h[1 + \log \frac{\sin \alpha}{\sin \beta}]}{\frac{1}{2}[\log \frac{1+\cos \beta}{1-\cos \beta} - \log \frac{1+\cos \alpha}{1-\cos \alpha}] - \log \frac{\sin \alpha}{\sin \beta}} \tag{3.4.1.28}$$

At $\phi = \beta$, m_ϕ must be continuous. Since at $\phi = \beta$, the yield regime changes from 5 to 8, this is possible if we have: $m_\phi(\beta^-) = m_\phi(\beta^+) = 0$. Also $\dot{w}$ must be continuous and since $|m_\phi(\beta)| \neq 1$, $\dot{w}'$ must be continuous as well.

The continuity of s and n_ϕ is not in question since they are expressed as continuous function of ϕ in the interval $(0,\alpha)$. The continuity of $\dot{v}$ has been used to find that $\dot{v} \equiv 0$ throughout the shell.

From $\dot{w}']_\beta = 0$, we have, with (3.4.1.18) and (3.4.1.27):

$$E = B \sin^2 \beta \qquad (3.4.1.29)$$

From $\dot{w}]_\beta = 0$, we have, with (3.4.1.18), (3.4.1.27) and (3.4.1.29):

$$C = B \cos \beta + \frac{1}{2} B \sin^2 \beta \left[\log \frac{1 + \cos \alpha}{1 - \cos \alpha} - \log \frac{1 + \cos \beta}{1 - \cos \beta}\right] \qquad (3.4.1.30)$$

With this result, (3.4.1.18) becomes:

$$\dot{w} = B \cos \beta - B \cos \phi + \frac{1}{2} B \sin^2 \beta \left[\log \frac{1 + \cos \alpha}{1 - \cos \alpha} - \log \frac{1 + \cos \beta}{1 - \cos \beta}\right]$$

At $\phi = 0$, we have:

$$\dot{w}(\phi=0) = \dot{w}_o = B \cos \beta - B + \frac{1}{2} B \sin^2 \beta \left[\log \frac{1 + \cos \alpha}{1 - \cos \alpha} - \log \frac{1 + \cos \beta}{1 - \cos \beta}\right]$$

or:

$$B = -\frac{\dot{w}_o}{1 - \cos\beta + \frac{1}{2}\sin^2\beta\,[\log\frac{1+\cos\beta}{1-\cos\beta} - \log\frac{1+\cos\alpha}{1-\cos\alpha}]} \qquad (3.4.1.31)$$

With this expression of B, (3.4.1.18) becomes:

$$\dot{w} = \dot{w}_o \frac{(\cos\phi - \cos\beta) + \frac{1}{2}\sin^2\beta\,[\log\frac{1+\cos\beta}{1-\cos\beta} - \log\frac{1+\cos\alpha}{1-\cos\alpha}]}{1 - \cos\beta + \frac{1}{2}\sin^2\beta\,[\log\frac{1+\cos\beta}{1-\cos\beta} - \log\frac{1+\cos\alpha}{1-\cos\alpha}]}$$

$$(0 \leq \phi \leq \beta) \qquad (3.4.1.32)$$

With E from (3.4.1.29) and B from (3.4.1.31), equation (3.4.1.27) becomes:

$$\dot{w} = \dot{w}_o \frac{\frac{1}{2}\sin^2\beta\,[\log\frac{1+\cos\phi}{1-\cos\phi} - \log\frac{1+\cos\alpha}{1-\cos\alpha}]}{1 - \cos\beta + \frac{1}{2}\sin^2\beta\,[\log\frac{1+\cos\beta}{1-\cos\beta} - \log\frac{1+\cos\alpha}{1-\cos\alpha}]}$$

$$(\beta \leq \phi \leq \alpha) \qquad (3.4.1.33)$$

It can be verified that with (3.4.1.32) and (3.4.1.33), $\dot{w}$ and $\dot{w}'$ are continuous at $\phi = \beta$.

From $m_\phi]_\beta = 0$, we have $m_\phi(\beta^-) = m_\phi(\beta^+) = 0$, or with (3.4.1.6), we derive:

$$p = p_s = 1 + h\,\frac{1 + \cos\beta}{1 - \cos\beta} \qquad (3.4.1.33a)$$

or

$$p_s = 1 + h\cot^2\beta/2 \qquad (3.4.1.33b)$$

By comparing (3.4.1.33a) above with (3.4.1.28), we have a relation to determine β when α is known and which can be

written as, after some reductions:

$$G(\beta) \equiv \frac{1}{2} (1 + \cos \beta)(\log \frac{1 + \cos \beta}{1 - \cos \beta} - \log \frac{1 + \cos \alpha}{1 - \cos \alpha})$$

$$- (1 - \cos \beta) - 2 \log \frac{\sin \alpha}{\sin \beta} = 0$$

(3.4.1.34)

It can be shown that:

$$G(0) = \lim_{\beta \to 0} [G(\beta)]$$

$$G(0) = 2 \log \frac{2}{\cos^2 \frac{\alpha}{2}} > 0, \text{ since } \cos^2 \frac{\alpha}{2} < 1$$

and that:

$$G(\alpha) = -(1 - \cos \alpha) < 0$$

Moreover the function $G(\beta)$ is continuous in the interval $(0,\alpha)$ and that its derivative is:

$$G'(\beta) = - \frac{1}{2} \sin \beta (\log \frac{1 + \cos \beta}{1 - \cos \beta} - \log \frac{1 + \cos \alpha}{1 - \cos \alpha})$$

$$- \frac{1 - \cos \beta}{\sin \beta} - \sin \beta$$

which is negative.

Thus $G(\beta)$ is continuous, decreasing monotonically in the interval $(0,\alpha)$ and $G(0) \times G(\alpha) < 0$.

Therefore there exists a unique value β such that:

$$G(\beta) = 0$$

For instance, with $\alpha = 90°$, it can be shown that $\beta \simeq 60°$.

With p_s from (3.4.1.33a), equations (3.4.1.4)-(3.4.1.7) become:

$$n_\phi = -\frac{1}{2} - h\,F(\phi) \qquad (0 \le \phi \le \alpha) \qquad (3.4.1.35)$$

$$n_\theta = -\frac{1}{2} + h\,F(\phi) \qquad (0 \le \phi \le \alpha) \qquad (3.4.1.36)$$

$$m_\phi = \frac{1}{2} - F(\phi) \qquad (0 \le \phi \le \beta) \qquad (3.4.1.37)$$

$$m_\theta = \frac{1}{2} + F(\phi) \qquad (0 \le \phi \le \beta) \qquad (3.4.1.38)$$

where

$$F(\phi) = \frac{1}{2}\,\frac{1 + \cos\beta}{1 - \cos\beta} \times \frac{1 - \cos\phi}{1 + \cos\phi} \qquad (3.4.1.39)$$

and (3.4.1.22) becomes:

$$m_\phi = \log\frac{\sin\phi}{\sin\beta} + \frac{1 + \cos\beta}{1 - \cos\beta}\left[\,\frac{1}{2}\left(\log\frac{1 + \cos\phi}{1 - \cos\phi} - \log\frac{1 + \cos\beta}{1 - \cos\beta}\right) + \log\frac{\sin\phi}{\sin\beta}\,\right]$$

$$(\beta \le \phi \le \alpha) \qquad (3.4.1.40)$$

Then:

$$m_\theta = 1 + \log\frac{\sin\phi}{\sin\beta} + \frac{1 + \cos\beta}{1 - \cos\beta}\left[\,\frac{1}{2}\left(\log\frac{1 + \cos\phi}{1 - \cos\phi} - \log\frac{1 + \cos\beta}{1 - \cos\beta}\right) + \log\frac{\sin\phi}{\sin\beta}\,\right]$$

$$(\beta \le \phi \le \alpha) \qquad (3.4.1.41)$$

The velocities are given by (3.4.1.26), (3.4.1.32) and (3.4.1.33).

2. Kinematic Admissibility

a) $0 \leq \phi \leq \beta$: regime 3-5 (Figs. 6a,b)

The conditions for kinematic admissibility in this interval are: $\dot{e}_\theta = \dot{e}_\phi < 0$, $\dot{k}_\theta = \dot{k}_\phi > 0$

From (3.1.1.1) and with $\dot{v}$ and $\dot{w}$ from (3.4.1.26) and (3.4.1.32), we have:

$$\dot{e}_\phi = \dot{e}_\theta =$$

$$- \dot{w}_o \frac{(\cos \phi - \cos \beta) + \frac{1}{2} \sin^2 \beta \, [\log\frac{1 + \cos \beta}{1 - \cos \beta} - \log\frac{1 + \cos \alpha}{1 - \cos \alpha}]}{1 - \cos \beta + \frac{1}{2} \sin^2 \beta \, [\log\frac{1 + \cos \beta}{1 - \cos \beta} - \log\frac{1 + \cos \alpha}{1 - \cos \alpha}]}$$

Since $\beta < \alpha$, we have $\log \frac{1 + \cos \beta}{1 - \cos \beta} - \log \frac{1 + \cos \alpha}{1 - \cos \alpha} > 0$, the function $\log \frac{1 + \cos x}{1 - \cos x}$ whose derivative is $- 2/\sin x$ being monotonically decreasing for $0 \leq x \leq \pi$.

Therefore $\dot{e}_\theta$ is negative since $\dot{w}_o > 0$.

From (3.1.1.2), and with (3.4.1.26) and (3.4.1.32), we have:

$$\dot{k}_\theta = \frac{h\dot{w}_o \cos \phi}{1 - \cos \beta + \frac{1}{2} \sin^2 \beta \, [\log \frac{1 + \cos \beta}{1 - \cos \beta} - \log \frac{1 + \cos \alpha}{1 - \cos \alpha}]}$$

$$(0 \leq \phi \leq \beta)$$

$\dot{k}_\theta$ is positive for $\phi \leq \beta \leq \pi/2$.

b) $\beta \leq \phi \leq \alpha$: regime 3-8 (Figs. 6a,b)

The conditions for kinematic admissibility in this

interval are: $\dot{e}_\theta = \dot{e}_\phi < 0$, $\dot{k}_\theta = -\dot{k}_\phi > 0$

With (3.4.1.26) and (3.4.1.33), we have:

$$\dot{e}_\theta = -\dot{w}_o \frac{\frac{1}{2} \sin^2 \beta \, [\log\frac{1 + \cos \phi}{1 - \cos \phi} - \log \frac{1 + \cos \alpha}{1 - \cos \alpha}]}{1 - \cos \beta + \frac{1}{2} \sin^2 \beta \, [\log \frac{1 + \cos \beta}{1 - \cos \beta} - \log \frac{1 + \cos \alpha}{1 - \cos \alpha}]}$$

$$(\beta \leq \phi \leq \alpha)$$

which is negative for $\beta \leq \phi \leq \alpha$.

Similarly:

$$\dot{k}_\theta = \frac{h \, \dot{w}_o \sin^2 \beta \cos \phi}{1 - \cos \beta + \frac{1}{2} \sin^2 \beta \, [\log \frac{1 + \cos \beta}{1 - \cos \beta} - \log \frac{1 + \cos \alpha}{1 - \cos \alpha}]}$$

$$(\beta \leq \phi \leq \alpha)$$

is positive for $\beta \leq \phi \leq \alpha \leq \pi/2$.

Thus the solution is kinematically admissible for $\alpha \leq \pi/2$.

3. Static Admissibility

The conditions for static admissibility are:

$$-1 \leq n_\theta \leq 0, \quad (0 \leq \phi \leq \alpha) \qquad (3.4.1.42)$$

$$-1 \leq n_\phi \leq 0, \quad (0 \leq \phi \leq \alpha) \qquad (3.4.1.43)$$

$$0 \leq m_\theta \leq 1, \quad (0 \leq \phi \leq \alpha) \qquad (3.4.1.44)$$

$$0 \leq m_\phi \leq 1, \quad (0 \leq \phi \leq \beta) \qquad (3.4.1.45)$$

$$-1 \leq m_\phi \leq 0, \quad (\beta \leq \phi \leq \alpha) \qquad (3.4.1.46)$$

From (3.4.1.39), we have:

$$F(\phi) = \frac{1}{2} \cot^2 \frac{\beta}{2} \tan^2 \frac{\phi}{2}$$

From this expression, we see that $F(\phi)$ increases with ϕ for $0 \leq \phi \leq \alpha \leq \pi/2$, and

$$F(\beta) = \frac{1}{2}$$

and the maximum of $F(\phi)$ is at $\phi = \alpha$:

$$F(\alpha) = \frac{1}{2} \cot^2 \frac{\beta}{2} \tan^2 \frac{\alpha}{2} \qquad (\alpha \leq \frac{\pi}{2})$$

Therefore, condition (3.4.1.45) is satisfied. For conditions (3.4.1.42), (3.4.1.43) to be satisfied, we must have:

$$h\ F(\alpha) \leq \frac{1}{2} \quad \text{or}$$

$$h \tan^2 \frac{\alpha}{2} \leq \tan^2 \frac{\beta}{2}$$

Since tan $\alpha/2$ and tan $\beta/2$ are of the same order, and h is very small compared to 1 for thin shell, this condition is expected to be satisfied, and therefore n_θ and n_ϕ are admissible.

For conditions (3.4.1.46), we have, from (3.4.1.40):

$$m'_\phi = - \frac{1 + \cos \beta - 2 \cos \phi}{\sin \phi} < 0$$

$$\text{for } \beta \leq \phi \leq \alpha$$

Thus, m_ϕ decreases monotonically from 0 to-1 as ϕ increases from β to α and is therefore admissible.

The admissibility of m_θ results consequently.

Therefore the solution is an exact one for the yield surface selected and the static collapse pressure of the clamped shell is given by (3.4.1.33a,b) or (3.4.1.28).

The solution is valid for $\alpha \leq \pi/2$.

If the base radius of the shell is taken as the reference length in the non-dimensionalizing procedure, it can be shown that with $\alpha^2 << 1$, expression (3.4.1.33a) reduces to the shallow shell case. (2.4.1.34)

3.4.2 Dynamic Response

A. If the peak value of the dynamic pressure is slightly above the static collapse pressure, the yield regime will be still 3-5 of Figs. (6a,b). The flow rules (3.4.1.14) and (3.4.1.15) remain valid, the boundary conditions remain the same, and therefore the velocity expressions (3.4.1.17) and (3.4.1.19) remain valid. We have then:

$$\dot{v} \equiv 0 \qquad (0 \leq \phi \leq \alpha) \qquad (3.4.1.17)$$

$$\dot{w} = \dot{w}_o \frac{\cos\phi - \cos\alpha}{1 - \cos\alpha} \qquad (3.4.1.19)$$

$$(0 \leq \phi \leq \alpha)$$

For regime 3-5, the relations (3.4.1.1) and (3.4.1.2) remain valid. With (3.4.1.1), (3.4.1.19) and (3.1.2.3), we obtain:

$$(s \sin \phi)' = \gamma \ddot{w}_o \frac{\sin \phi \cos \phi - \cos \alpha \sin \phi}{1 - \cos \alpha} - (p-1) \sin \phi$$

Integrating from 0 to ϕ, we obtain:

$$s \sin \phi = \frac{\gamma \ddot{w}_o}{1 - \cos \alpha} [\frac{1}{2} \sin^2 \phi - \cos \alpha (1 - \cos \phi)]$$

$$- (p-1) (1 - \cos \phi) \qquad (3.4.2.1)$$

With relation (3.4.1.2), and equation (3.1.2.1), we have:

$$h[m_\phi' \sin \phi + 2m_\phi \cos \phi] = h \cos \phi + s \sin \phi$$

Multiplying both sides by $\sin \phi$, and with $s \sin \phi$ from (3.4.2.1), we obtain:

$$h \frac{d}{d\phi} (m_\phi \sin^2 \phi) = h \cos \phi \sin \phi$$

$$+ \frac{\gamma \ddot{w}_o}{1 - \cos \alpha} [\frac{1}{2} \sin^3 \phi - \cos \alpha (1 - \cos \phi) \sin \phi]$$

$$- (p - 1)(1 - \cos \phi) \sin \phi$$

Integrating from 0 to ϕ and simplifying by $\sin^2 \phi$, we obtain, after some reductions:

$$m_\phi = \frac{1}{2} + \frac{\gamma \ddot{w}_o}{h(1-\cos\alpha)} \times \frac{1}{1+\cos\phi}[\frac{1}{2} - \frac{1+\cos\phi + \cos^2\phi}{6} - \frac{\cos\alpha}{2}(1-\cos\phi)]$$

$$- \frac{p-1}{2h} \frac{1-\cos\phi}{1+\cos\phi} \qquad (3.4.2.2)$$

At $\phi = \alpha$, $m_\phi = 0$, from which we derive:

$$\gamma\ddot{w}_o = \frac{3}{2}\left(p - 1 - h\,\frac{1 + \cos\alpha}{1 - \cos\alpha}\right)$$

With p_s given by (3.4.1.8a), we have:

$$\gamma\ddot{w}_o = \frac{3}{2}(p - p_s). \qquad (3.4.2.3)$$

With this result, equation (3.4.2.1) becomes, after some rearrangements:

$$s \sin\phi = \frac{3}{2}\,\frac{p - p_s}{1 - \cos\alpha}\,(1 - \cos\phi)\left[\frac{1 + \cos\phi}{2} - \cos\alpha\right] - (p - 1)(1 - \cos\phi).$$

Using again p_s from (3.4.1.8a), we have:

$$1 = p_s - h\,\frac{1 + \cos\alpha}{1 - \cos\alpha}$$

From this result, we get:

$$s \sin\phi = -\frac{p - p_s}{4}\,\frac{1 - \cos\phi}{1 - \cos\alpha}\,(1 + 2\cos\alpha - 3\cos\phi) - h\,\frac{1 + \cos\alpha}{1 - \cos\alpha}\,(1 - \cos\phi) \qquad (3.4.2.4)$$

Similarly, equation (3.4.2.2) becomes, after some reductions:

$$m_\phi = \frac{1}{2} - \frac{1 + \cos\alpha}{1 - \cos\alpha} \times \frac{1 - \cos\phi}{1 + \cos\phi} + \frac{p - p_s}{4h}\,\frac{1 - \cos\phi}{1 - \cos\alpha}\,\frac{\cos\phi - \cos\alpha}{1 + \cos\phi}$$

(3.4.2.5)

With $\ddot{v} = 0$ and using relation (3.4.1.1), we have, from equation (3.1.2.2):

$$n'_\phi \sin\phi + 2n_\phi \cos\phi = -\cos\phi + s\sin\phi$$

Multiplying both sides by $\sin\phi$, and with $s\sin\phi$ from (3.4.2.4), we obtain:

$$\frac{d}{d\phi}(n_\phi \sin^2\phi) = -\sin\phi\cos\phi$$

$$-\frac{p-p_s}{4}\,\frac{1-\cos\phi}{1-\cos\alpha}\,(\sin\phi + 2\cos\alpha\sin\phi - 3\sin\phi\cos\phi)$$

$$-h\,\frac{1+\cos\alpha}{1-\cos\alpha}(\sin\phi - \sin\phi\cos\phi)$$

Integrating from 0 to ϕ, and simplifying by $\sin^2\phi$, we obtain after some reductions:

$$n_\phi = -\frac{1}{2} - \frac{h}{2}\,\frac{1+\cos\alpha}{1-\cos\alpha} \times \frac{1-\cos\phi}{1+\cos\phi} + \frac{p-p_s}{4}\,\frac{1-\cos\phi}{1-\cos\alpha}\,\frac{\cos\phi-\cos\alpha}{1+\cos\phi} \qquad (3.4.2.6)$$

Comparing (3.4.2.6) above with (3.4.2.5), we see that we can write these expressions in the following terms:

$$m_\phi = \frac{1}{2} - F(\cos\phi) \qquad (3.4.2.7)$$

$$n_\phi = -\frac{1}{2} - hF(\cos\phi) \qquad (3.4.2.8)$$

with $F(\cos\phi)$ now defined as:

$$F(\cos\phi) = \frac{1}{2}\,\frac{1+\cos\alpha}{1-\cos\alpha} \times \frac{1-\cos\phi}{1+\cos\phi} - \frac{p-p_s}{4h}\,\frac{1-\cos\phi}{1-\cos\alpha}\quad\frac{\cos\phi-\cos\alpha}{1+\cos\phi} \qquad (3.4.2.9)$$

Then with relations (3.4.1.1) and (3.4.1.2), we obtain

$$m_\theta = \frac{1}{2} + F(\cos\phi) \qquad (3.4.2.10)$$

$$n_\theta = -\frac{1}{2} + hF(\cos\phi) \qquad (3.4.2.11)$$

From (3.4.2.3), we obtain:

$$\gamma\dot{w}_o = \frac{3}{2}\,[p_o(1-e^{-\tau}) - p_s\tau] \qquad (3.4.2.12)$$

With this result, the velocities are given by (3.4.1.17) and (3.4.1.19).

Admissibility

1. Kinematic

From the solution of the static problem section 3.4.1 we have found that the solution is kinematically admissible for

$$\alpha \leq \pi/2 \qquad \text{and}$$

$$\dot{w}_o \geq 0$$

With $\dot{w}_o$ from (3.4.2.12), we see that $\dot{w}_o$ will be positive for $0 \leq \tau \leq \tau_f$, where τ_f is determined by:

$$p_o(1-e^{-\tau_f}) - p_s\tau_f = 0 \qquad (3.4.2.13)$$

At $\tau = \tau_f$, $\dot{w} = 0$ and the motion ceases.

2. Static

For regime 3-5, Figs. (6a,b), the conditions for static admissibility are:

$$-1 \leq n_\phi \leq 0 \quad , \quad -1 \leq n_\theta \leq 0$$

$$0 \leq m_\phi \leq 1 \quad , \quad 0 \leq m_\theta \leq 1$$

From the expressions of m_ϕ, n_ϕ, m_θ, n_θ in (3.4.2.7-(3.4.2.11) and since h is small compared to 1, we see that the necessary and sufficient condition for these inequalities to be satisfied is:

$$-\frac{1}{2} \leq F(\cos\phi) \leq \frac{1}{2} \qquad (3.4.2.14)$$

where $F(\cos\phi)$ has been defined in (3.4.2.9).

It can be shown that:

if $p-p_s > 0$, or: $0 \leq \tau \leq \log \frac{p_o}{p_s} = \tau_\ell$,

condition (3.4.2.14) will be satisfied if:

$$p_o-p_s \leq 4h \frac{[1+\sqrt{1+\cos\alpha}]^2}{1-\cos\alpha} \qquad (3.4.2.15)$$

if $p-p_s < 0$, or: $\tau_\ell \leq \tau \leq \tau_f$,

condition (3.4.2.14) will be satisfied if:

$$p_s-p_o e^{-\tau_f} \leq \frac{4h}{1-\cos\alpha} \qquad (3.4.2.16)$$

Therefore the solution is kinematically admissible if the peak value p_o of the dynamic pressure satisfies (3.4.2.15) and (3.4.2.16). The lesser of the maximum values of p_o obtained from (3.4.2.15) and (3.4.2.16) will

give the upper limit p_{oL} of the peak value p_o. If the shell is not very shallow it has been found that condition (3.4.2.16) is much more restrictive than (3.4.2.15). Then the value p_{oL} can be obtained from the system made of (3.4.2.13) and the equality (3.4.2.16):

$$(S)\begin{cases} p_{oL}(1-e^{-\tau_f}) - p_s\tau_f = 0 & (3.4.2.13) \\ p_s - p_{oL}e^{-\tau_f} = \dfrac{4h}{1 - \cos\alpha} & (3.4.2.16) \end{cases}$$

p_s is given from (3.4.1.8a,b).

C. Conclusion

For $p_s \leq p_o \leq p_{oL}$

where p_{oL} is determined by the system (S) above, the solution is statically and kinematically admissible.

The stresses and velocities are given by (3.4.2.7)-(3.4.2.12) and (3.4.1.17) (3.4.1.19).

The final displacement distribution is obtained by integrating the velocities from 0 to τ_f:

$$\gamma v_f = 0$$

$$\gamma w_f = \gamma w_o \frac{\cos\phi - \cos\alpha}{1 - \cos\alpha} \qquad (3.4.2.17)$$

where γw_o is the final central displacement which is:

$$\gamma w_o = \frac{3}{2}\left[(p_o - p_s)\tau_f - \frac{1}{2}p_s\tau_f^2\right] \qquad (3.4.2.18)$$

where τ_f has been defined in (3.4.2.13) and which is the instant when the motion ceases.

The energy absorbed is:

$$E_{abs} = \int_0^{\tau_f} \int_0^{\alpha} P \frac{dU_n}{dt} \times 2\pi R^2 \sin\phi \, d\phi \, dt$$

or, in terms of non-dimensional quantities:

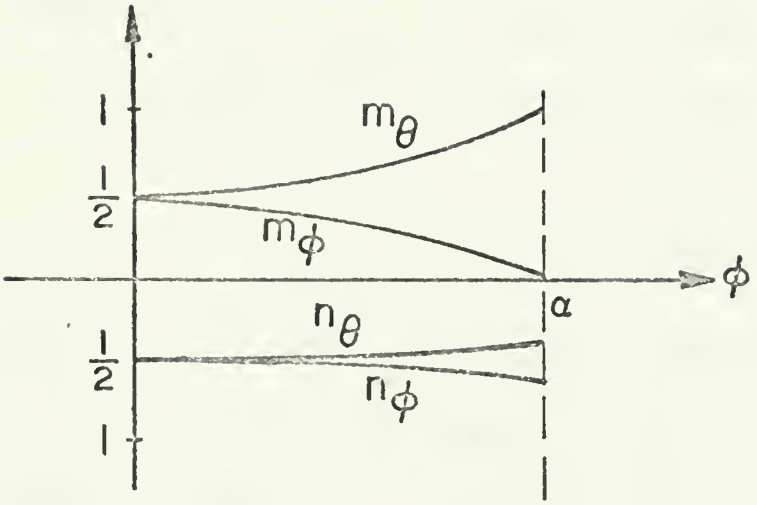
$$P = \frac{N_o}{R} p$$

$$\frac{d}{dt} (U_n) = R \frac{\dot{w}}{T_o} , \quad dt = T_o d\tau :$$

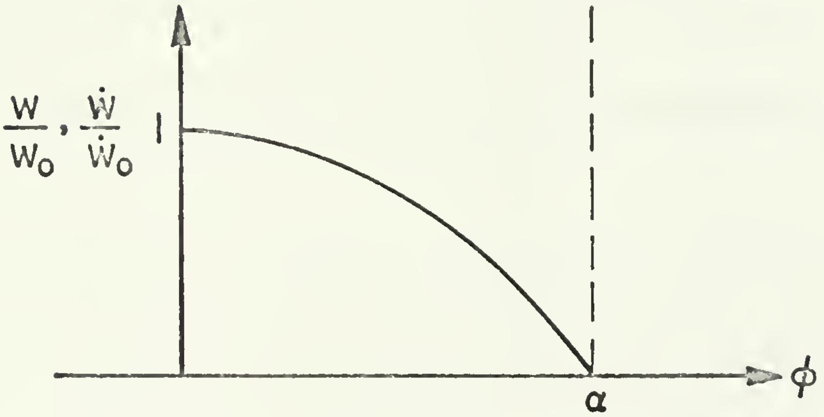
$$E_{abs} = 2\pi R^2 N_o \int_0^{\tau_f} \int_0^{\alpha} p \, \dot{w} \sin\phi \, d\phi \, d\tau$$

with $\dot{w}$ from (3.4.1.19), we obtain:

$$E_{abs} = 2\pi R^2 N_o \frac{3(1 - \cos\alpha)}{4\gamma} \left\{ \frac{p_o^2}{2} (1 - e^{-\tau_f})^2 - p_o p_s [1 - (1 + \tau_f) e^{-\tau_f}] \right\} \qquad (3.4.2.19)$$



(a) Stress Distribution



(b) Velocity and Displacement Distribution

Stress and Velocity, Displacement Distributions
Deep Shell - Uncoupled Diamond Yield Surface

Fig. 22

3.5 Comparison of the Results

3.5.1 Two-Moment Limited Interaction Yield Surface

From the results presented in (3.2.2.1)-(3.2.2.7) we see that the solution obtained is rather complicated, even in the low pressure range where the whole shell yield in one regime.

The limited series expansion solution (3.2.3.2)-(3.2.3.6), neglecting terms of 4th order and higher, could give a better indication on the upper limit of the peak value of the dynamic pressure. Although the problem of static admissibility has to resort to numerical method, this solution gives an indication on the maximum depth of the shell when the dynamic load is a rectangular pulse, in (3.2.3.16).

An analysis for a higher range of pressure was attempted,by using a series solution, the results obtained become too complicated, the algebra too involved and no meaningful conclusions could be reached. We did find, however, that at higher pressure range, the shell would yield in at least 3 regimes like the medium pressure case of shallow shells. The solution which assumed the shell yielding under 2 regimes arrived at the same conclusion as in the case of shallow shells: when ϕ/α reaches a value of about 0.8, $\dot{e}_\phi$ becomes positive and the solution is no longer kinematically admissible.

The upper limit of p_o in the low pressure case (3.2.3.10) is comparable to that of the shallow shell, (2.2.2.1.21), together with the fact that the shell does not yield under 2 regimes at higher pressure and other similarities found in the admissibility considerations suggest that there is a strong analogy in the solutions of deep and shallow shells, and that any further development of the deep spherical shells could be made by adopting tentative solutions which are constituted of the same yield regimes as in the shallow shell solutions. Obviously the choice of yield regimes does not solve the whole problem but it gives some help to start the solution.

3.5.2 Uncoupled Diamond Yield Surface

The solution obtained (3.4.2.7)-(3.4.2.12), (3.4.1.17), (3.4.1.19) are simple enough so that the whole problem: obtainment of a solution and examination of the admissibility of this solution, can be solved analytically for shallow shells as well as for deep shells, the generalized stresses are expressible in terms of a single function whose variations could be obtained without much difficulty. The membrane stresses are small compared to the moment resultants, and it appears as though they were "trapped" within face 3 (Figs. 6a,b). The meridional displacement is zero, the normal displacement has a cosine distribution,

the maximum being at the apex which is also a regular point for the displacement function (no discontinuity in slope).

4. CONCLUSIONS

By using the two-moment limited interaction yield surface proposed by Hodge [18] and the shallow shell approximations, we have obtained the solution for the dynamic response of shallow spherical shells for medium high pressures which satisfy the inequality $6h \leq p_o - p_s \leq \lambda h$ where λ depends on the ratio r = depth/thickness of the shell (e.g., for $r = 1$, $\lambda = 19.1$, for $r = 5$, $\lambda = 13.2$). We have seen that there are three stages of motion: in the first stage of motion ($0 \leq \tau \leq \tau_o$), the shell yields under 4 regimes, in the second stage ($\tau_o \leq \tau \leq \tau_1$) under 3 regimes and in the third stage ($\tau_1 \leq \tau \leq \tau_f$) under 1 regime. The solution obtained is rather complicated. The restriction on the upper limit of the peak value of the dynamic pressure for shells whose ratio depth/thickness is larger than 3/8 arises because the moment resultant m_ϕ must remain positive throughout the duration of the motion. This requirement is most critical at the instant when the shell comes to rest.

A first attempt to increase the upper limit on pressure would be to allow m_ϕ to be negative at some instant during the third stage of motion ($\tau_1 \leq \tau \leq \tau_f$). Then from this instant the shell would collapse under 2 regimes: probably 5-7 and 5-β and the subsequent behavior probably would not return to one regime again, since regime 5-β would most likely be spreading towards the center of the shell with

time. If this problem can be solved successfully, the increase of this upper limit would not be likely to be important since sooner or later, and sooner rather than later, n_θ would become smaller than -1 near the outer edge. Then at least another regime will have to be added to the collapse pattern of the shell, so that from this instant, the shell would collapse under 3 regimes, separated by 2 boundary circles, probably shrinking towards the center, and catching up or widening the gap between each other. Still this upper limit of the peak value of the dynamic pressure would be limited by the fact that this pattern may occur only during the third stage of motion ($\tau_1 \leq \tau \leq \tau_f$) when the shell starts to collapse in one regime.

Thus the complete solution for shallow shells of practical geometry (ratio depth/thickness larger than 3/8) subjected to any magnitude of pressure would be the one that starts with at least 6 regimes separated by 5 boundary circles all moving towards the center. The mathematics would be complicated.

For deeper shells, the problem is still more involved and only a solution for low pressure range has been obtained. The limited series expansion of this solution has allowed us to gain a better understanding of the upper limit of the peak value of the dynamic pressure and to draw some analogies between solutions for deep and shallow shells, and thus

to decide on the direction to start a solution for higher pressure range if one has the desire.

One noteworthy aspect of the solution for the shallow shell according to the two-moment limited interaction yield surface is that the meridional component of the velocity is of the order of $0(Z/R)$ of its normal component so that it cannot be neglected, since the approximation is that only $(Z/R)^2$ is negligible with respect to 1. Another aspect concerns the variations of n_θ, which are quite small so that n_θ remains very close to -1, and which may be a source of trouble if one desires to go to higher pressures.

With the uncoupled diamond yield surface proposed by Jones, we have seen that for spherical shells, the meridional component of the velocity is zero, and n_θ and n_ϕ have been brought within a narrow interval around the midpoint of the corresponding yield regime. The solutions obtained, for shallow and deep shells, are quite simple and the problem of admissibility, which is usually as difficult as obtaining the solution itself in the case of the two-moment limited interaction yield surface, can be examined without much difficulty. It is unfortunate that attempts to obtain solutions for higher pressures did not succeed.

We should note, however, that the uncoupled diamond yield surface does not monopolize in denying us a solution. In the limit analysis of spherical caps with the two-moment

limited interaction yield surface [18], the cap angle has been limited to a value smaller than 90°; otherwise the value of n_θ at α would become positive. We have tried to extend this solution to higher value of the cap angle by assuming the following regimes:

For $0 \leq \phi \leq \beta$: regime 5-7 (Figs. 4a,b)
For $\beta \leq \phi \leq \alpha$: regime 6-7 (Figs. 4a,b)

At $\phi = \beta$, we obtain:

$$n_\theta(\beta^-) = 0 \quad \text{and}$$
$$n_\theta(\beta^+) = 0$$

The first equality comes as a definition of the angle β and the second as a consequence of the continuity of n_ϕ. Thus in the interval (β,α) n_θ should have a minimum at β. We have found that at $\phi = \beta^+$, $n'_\theta = 0$ and $n''_\theta = -\frac{2}{\sin^2\beta} < 0$, resulting rather in a maximum at $\phi = \beta^+$, which is not possible.

Other combinations made of regime 5-7, 6-7 and 6-β were also not successful.

The common factor between the Hodge's and Jones' yield surfaces is the absence of interaction between moments and membrane stresses. Perhaps for general shells made of rigid-perfectly plastic material, the interaction between moments and membrane stresses would not be negligible and more sophisticated yield surfaces should be used.

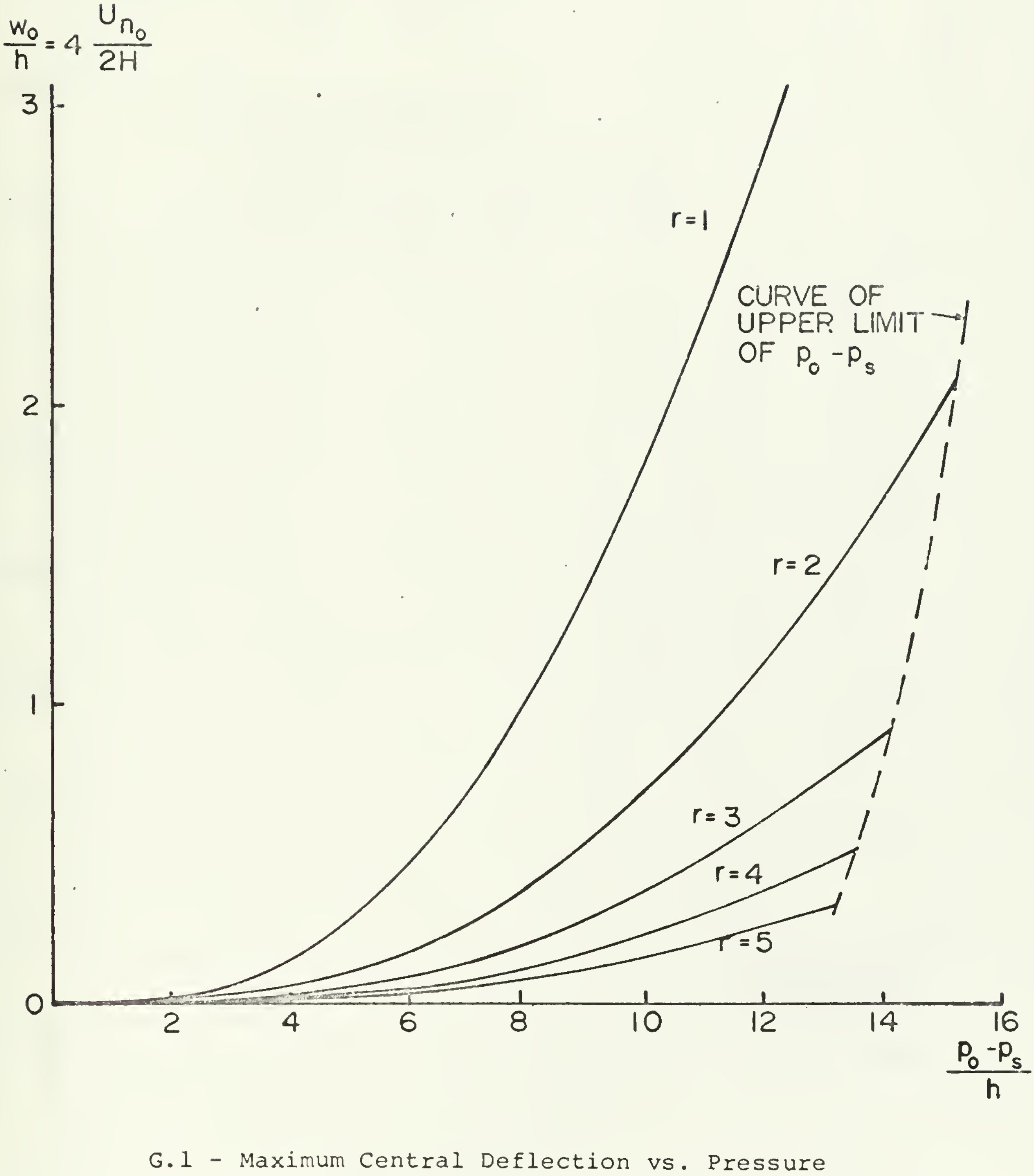
In the case of static plasticity, the general theorems allow us to construct bounds on the collapse load obtained from an approximate yield surface. In dynamic plasticity, no such general theorems exist, although Hodge and Paul [13] gave some useful hints in that direction.

From this study, we could see that the problem of dynamic plasticity of spherical shells loaded symmetrically, even with the use of approximate yield surfaces, is rather complicated, and a complete analytical solution for a wide range of pressures is not likely to be possible in the near future, if at all.

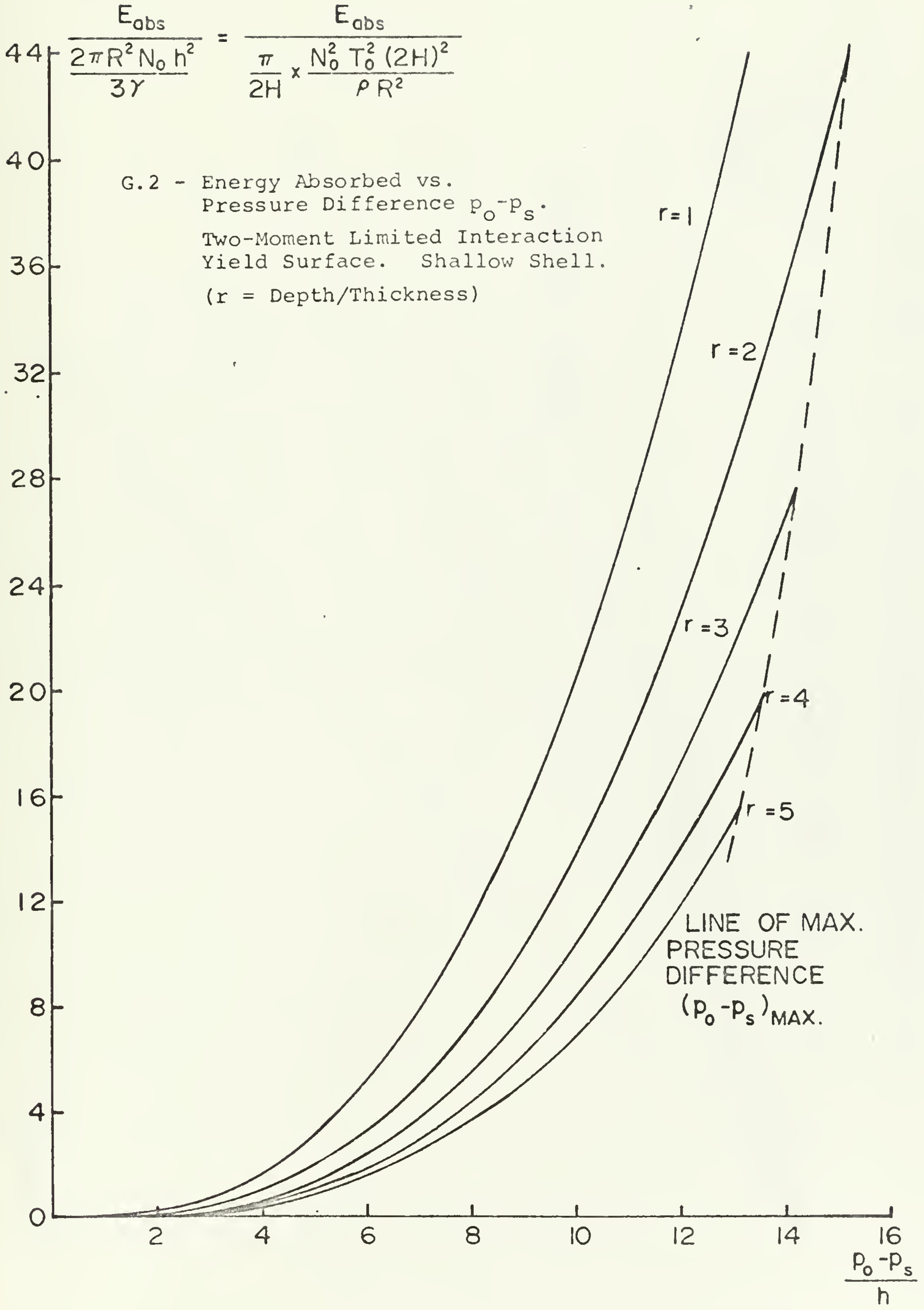
It might be mentioned that the only previous studies on the rigid-plastic behavior of shallow shells are those of Hodge and Lakshmikantham [23] who obtained lower and upper bounds to the static collapse pressure of shallow parabolic and conical caps with central holes.

It is hopeful that the analytical solutions presented herein for the dynamic and static behavior of shells would be useful in checking the validity and accuracy of numerical programmes and to assist in the development of approximate methods such as those developed in [24]. It appears that a numerical procedure would have to be used in order to extend the existing solution to higher pressures. Then, relatively more complicated yield surfaces can be used, such as the one proposed by W. Flügge and Tsuneyoshi Nakamura [22].

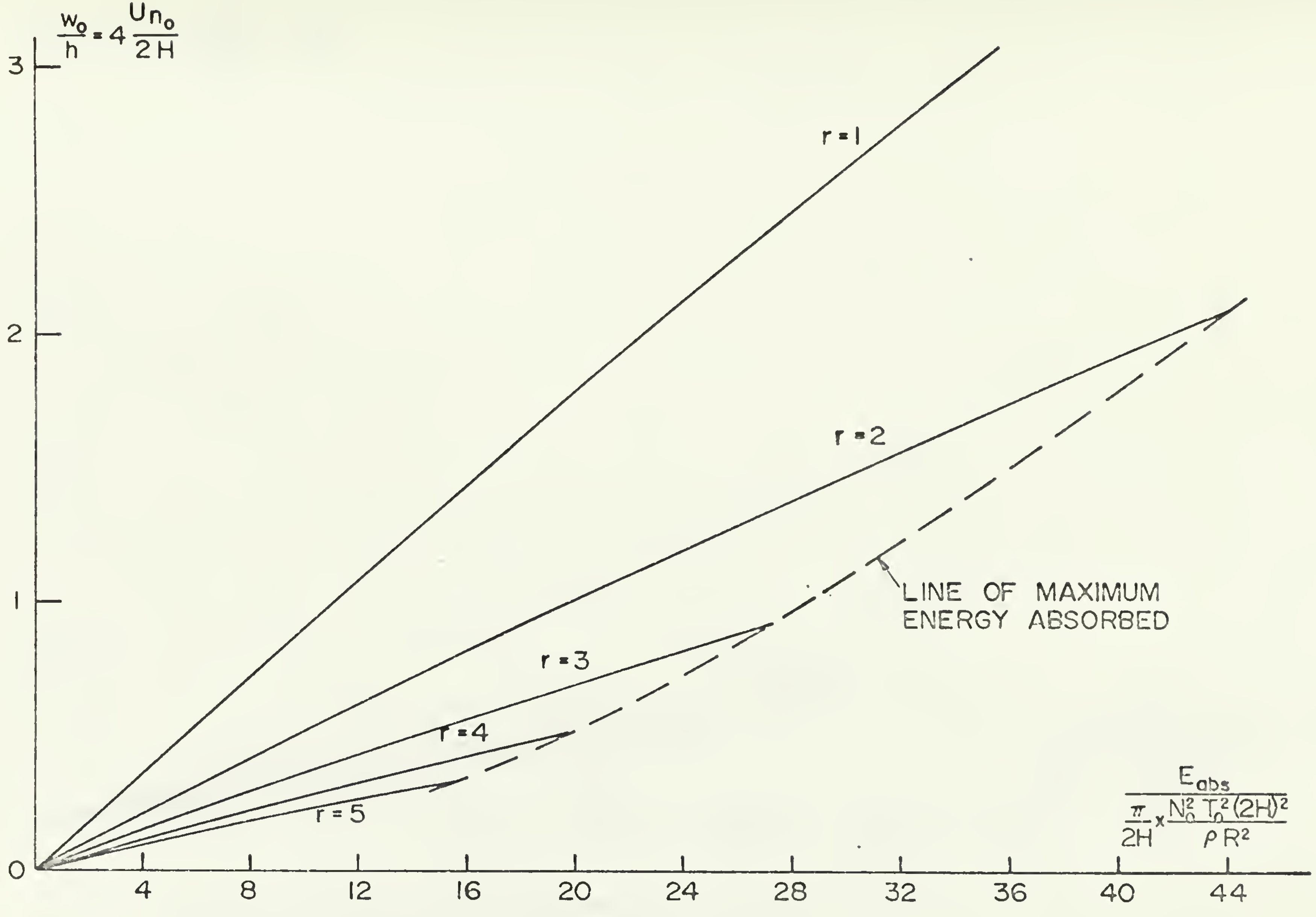
Thus, it is believed that numerical methods and more sophisticated yield surfaces would be the direction for future researches on dynamic behavior of shells of revolution.



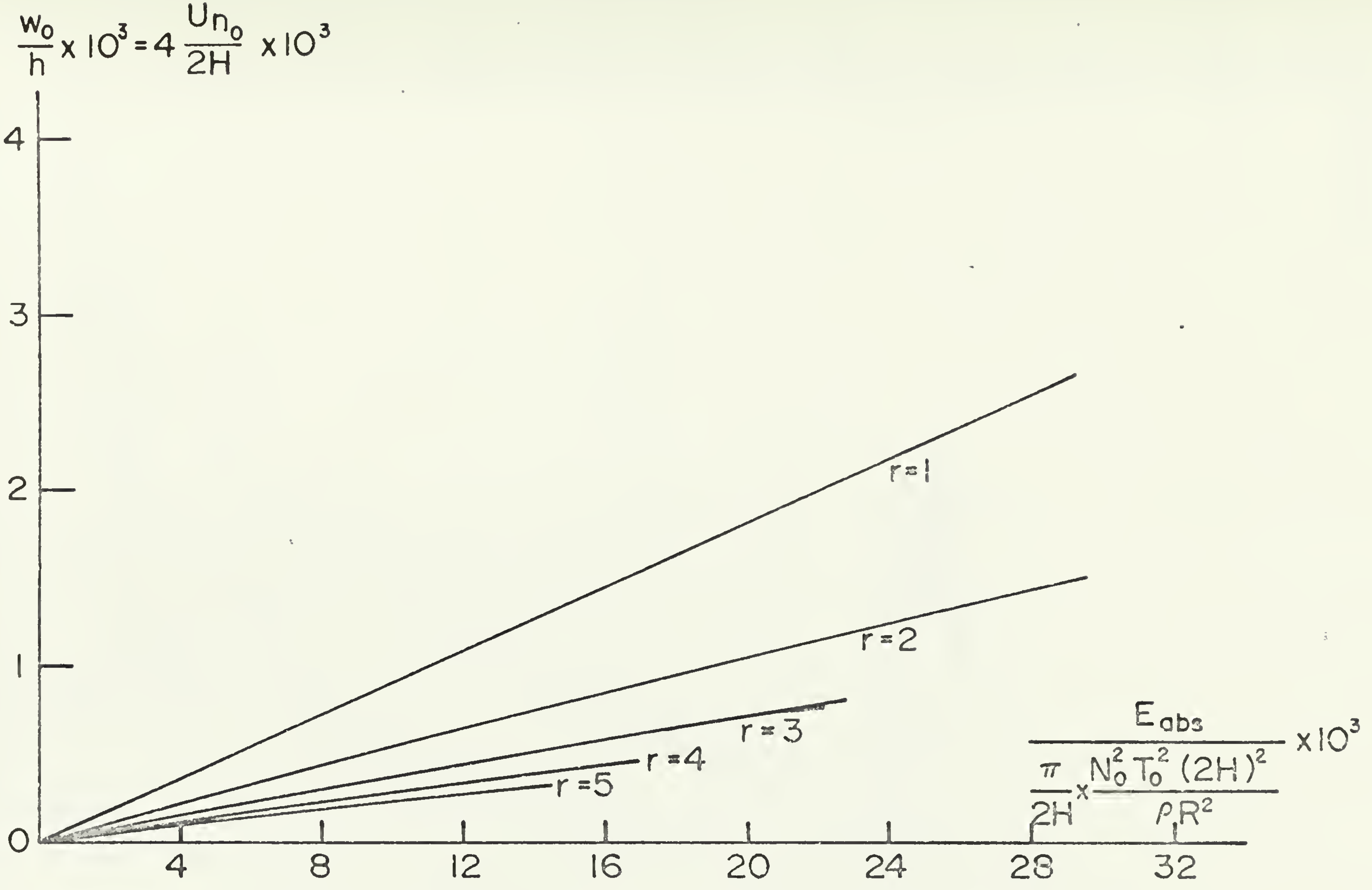
G.1 - Maximum Central Deflection vs. Pressure Difference p_o-p_s. Shallow Shell.
Two Moment Limited Interaction Yield Surface (r = Depth/Thickness)



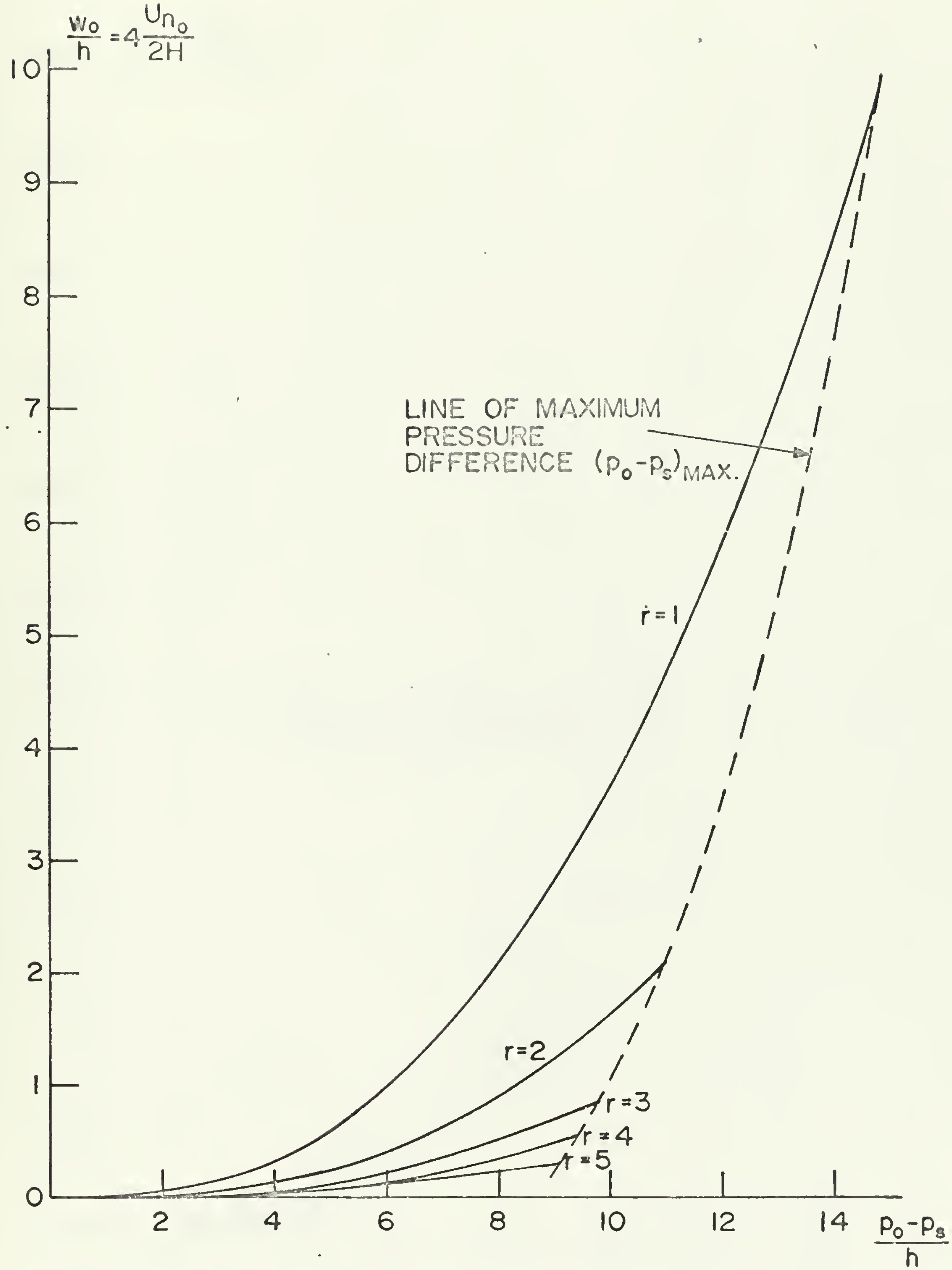
G.2 - Energy Absorbed vs. Pressure Difference $p_o - p_s$. Two-Moment Limited Interaction Yield Surface. Shallow Shell. (r = Depth/Thickness)



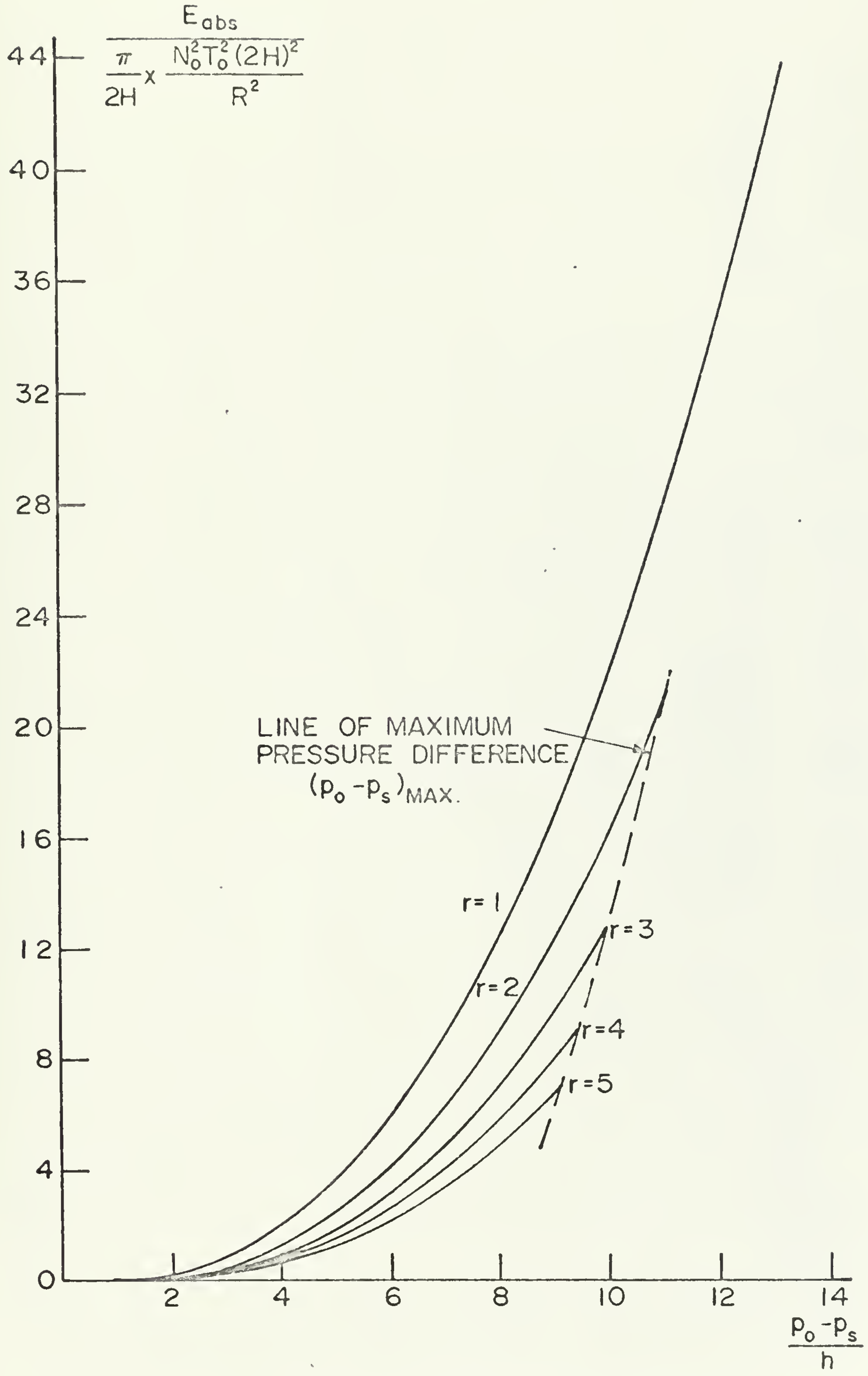
G.3a - Maximum Central Deflection vs. Energy Absorbed. Shallow Shell. Two-Moment Limited Interaction Yield Surface. (r = Depth/Thickness)



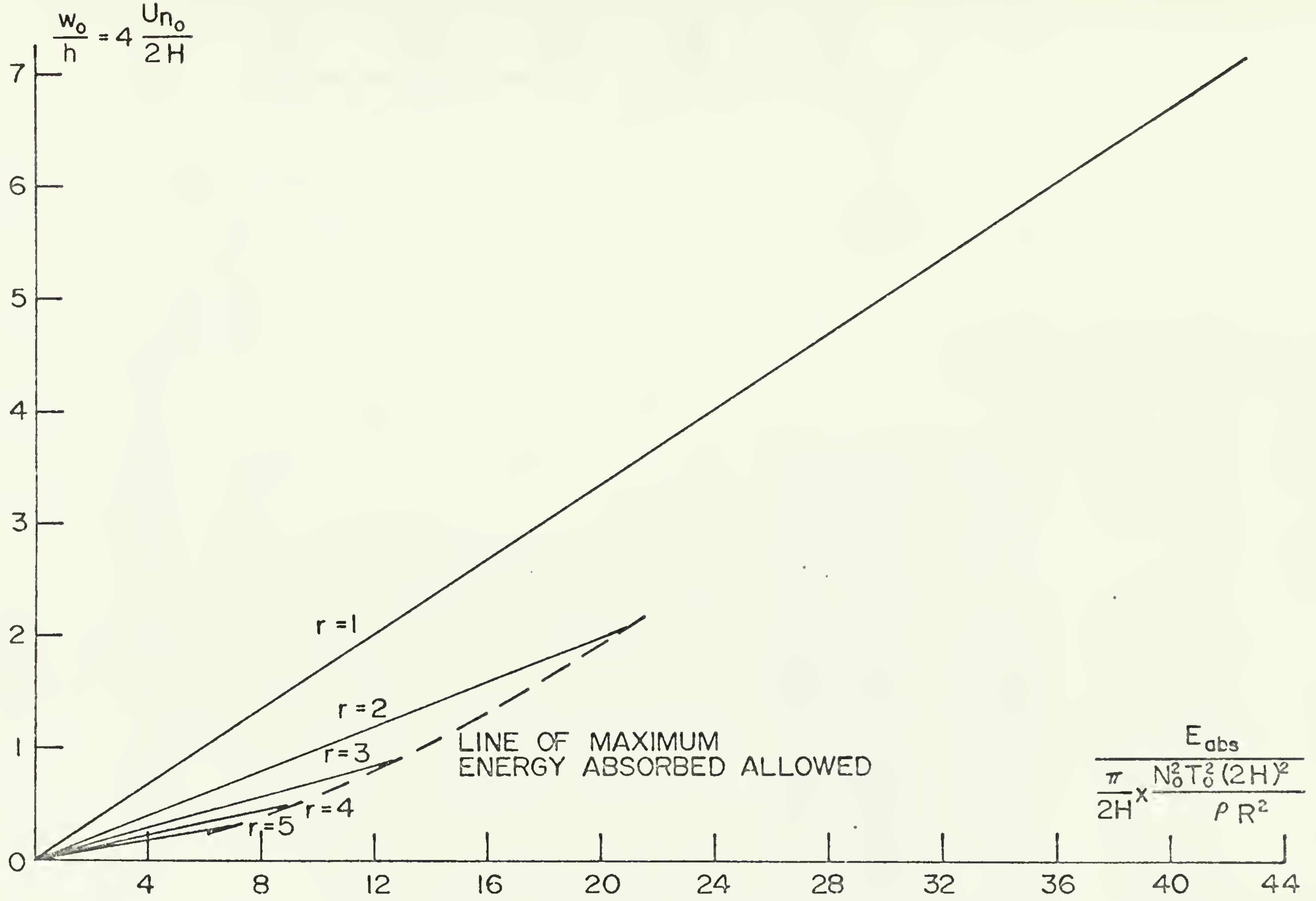
G.3b - Maximum Central Deflection vs. Energy Absorbed at Very Low Pressure Difference $p_o - p_s$.
Two-moment limited interaction yield surface.
Shallow Shell.



G.4 - Maximum Central Deflection vs. Pressure Difference $p_o - p_s$. Uncoupled Diamond Yield Surface. (r=depth/thickness) Shallow Shell.



G.5 - Energy Absorbed vs. Pressure Difference $p_o - p_s$.
Uncoupled Diamond Yield Surface.
(r = Depth/Thickness) Shallow Shell.



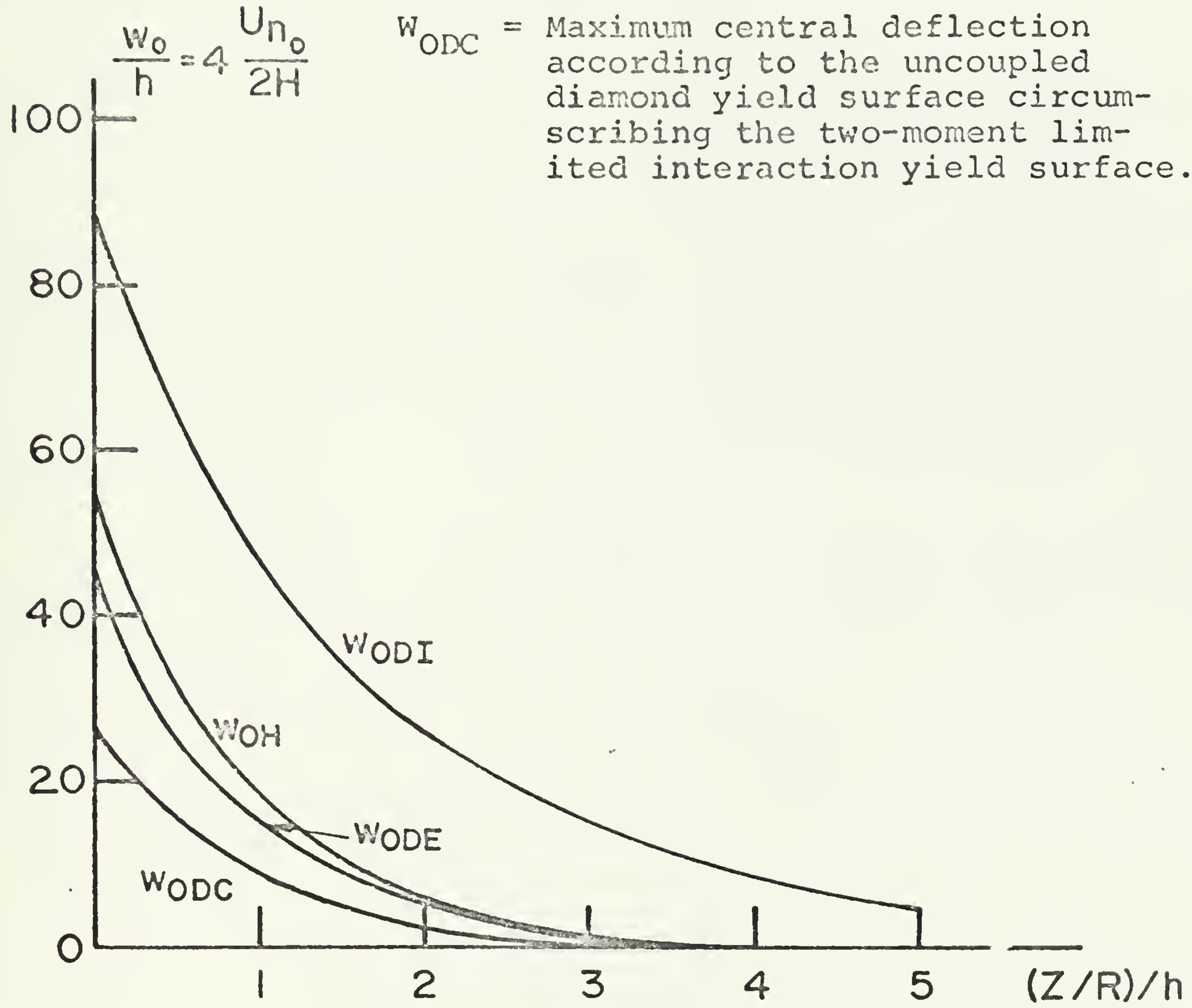
G.6 - Maximum Central Deflection vs. Energy Absorbed. Shallow Shell. Uncoupled Diamond Yield Surface. (r = Depth/Thickness)

W_{OH} = Maximum central deflection according to the two-moment limited interaction yield surface.

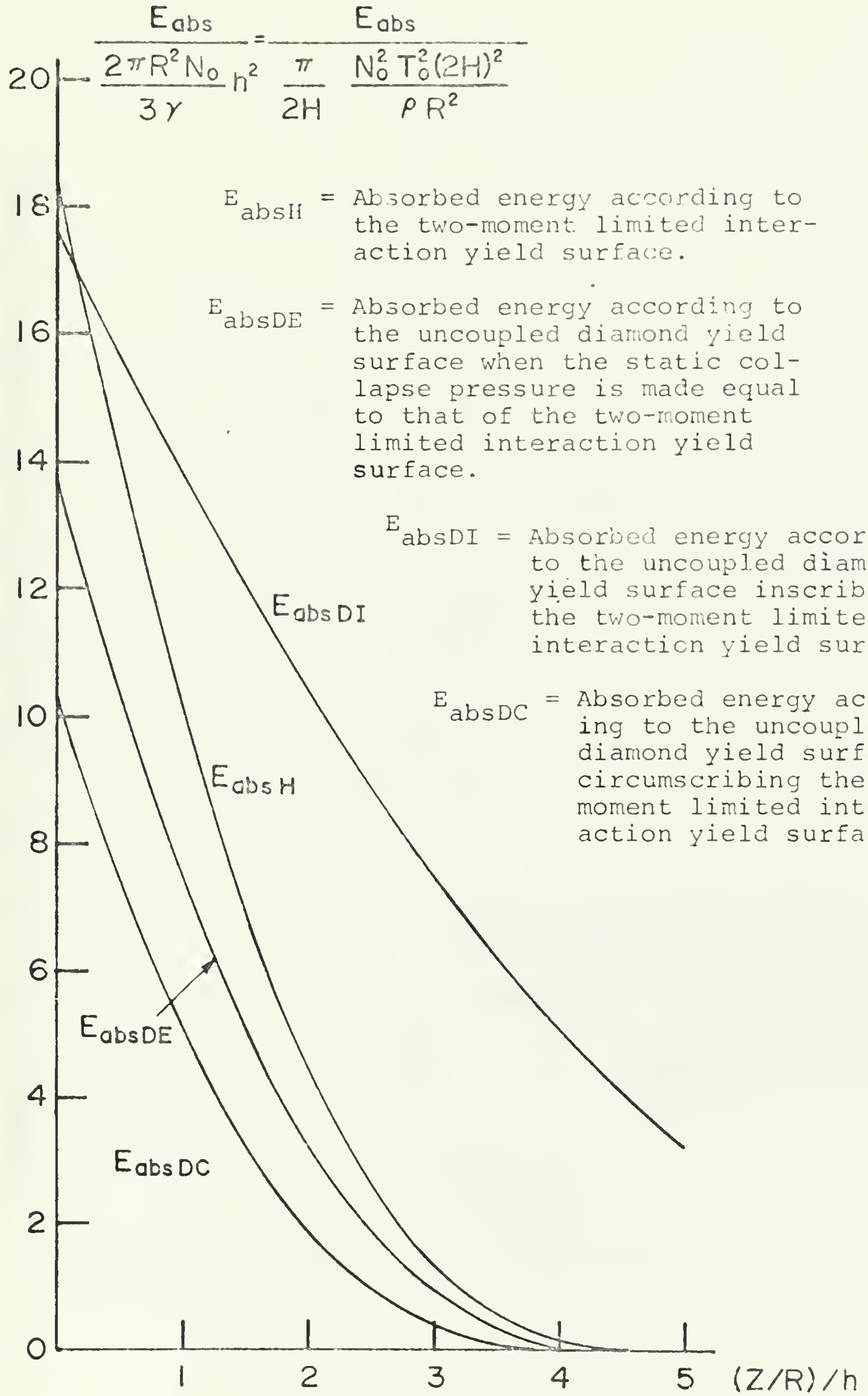
W_{ODE} = Maximum central deflectin according to the uncoupled diamond yield surface when the static collapse pressure is made to be equal to that of the two-moment limited interaction surface.

W_{ODI} = Maximum central deflection according to the uncoupled diamond yield surface inscribing the two-moment limited interaction yield surface.

W_{ODC} = Maximum central deflection according to the uncoupled diamond yield surface circumscribing the two-moment limited interaction yield surface.



G.7 - Compared Maximum Central Deflection ($p_o = 26h$) Shallow Shell.



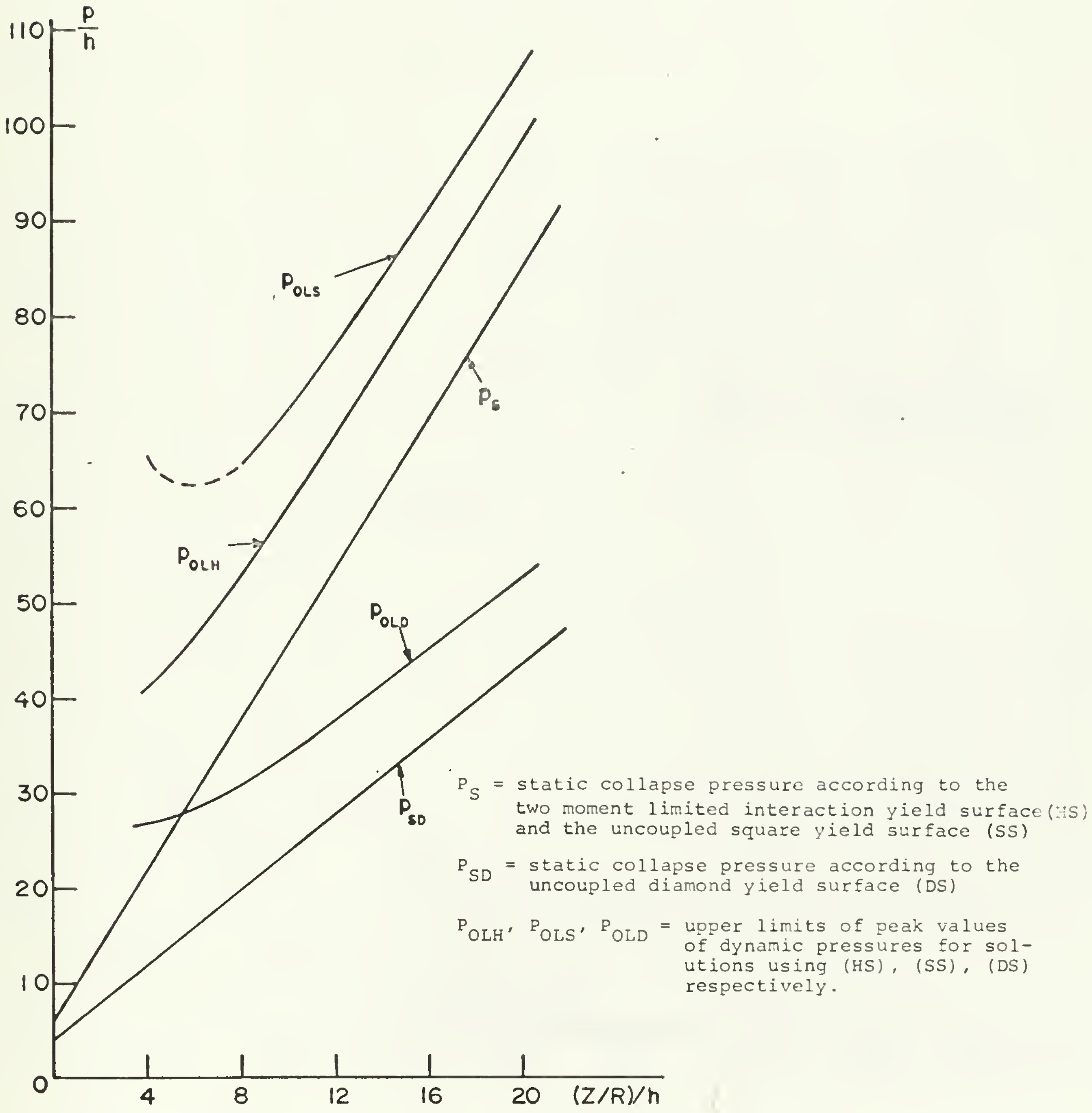
E_{absH} = Absorbed energy according to the two-moment limited interaction yield surface.

E_{absDE} = Absorbed energy according to the uncoupled diamond yield surface when the static collapse pressure is made equal to that of the two-moment limited interaction yield surface.

E_{absDI} = Absorbed energy according to the uncoupled diamond yield surface inscribing the two-moment limited interaction yield surface.

E_{absDC} = Absorbed energy according to the uncoupled diamond yield surface circumscribing the two-moment limited interaction yield surface.

G.8 - Compared Absorbed Energy (p_o = 26h)
Shallow Shell.



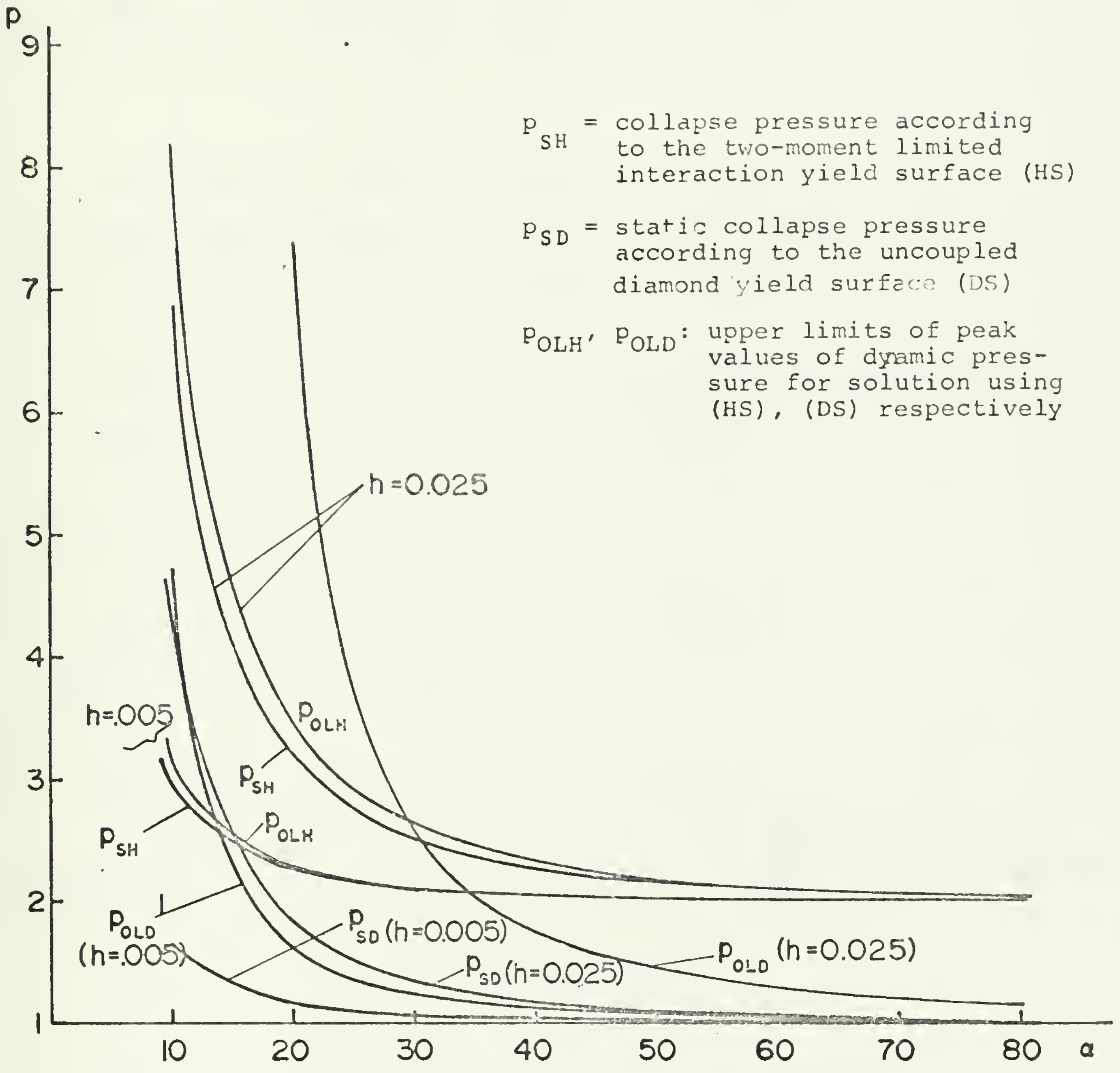
P_S = static collapse pressure according to the two moment limited interaction yield surface (HS) and the uncoupled square yield surface (SS)

P_{SD} = static collapse pressure according to the uncoupled diamond yield surface (DS)

P_{OLH}, P_{OLS}, P_{OLD} = upper limits of peak values of dynamic pressures for solutions using (HS), (SS), (DS) respectively.

G.9 - Static Collapse Pressure for Simply Supported Shells and Upper Limits of the Peak Values of Dynamic Pressure

(Shallow Shell)



G.10 - Static Collapse Pressure for Simply Supported Shells and Upper Limits of the Peak Values of Dynamic Pressure (Deep Shell)

APPENDIX A1

MEDIUM PRESSURE

$1.2h \leq p_o - p_s \leq 6.0h$

Determination of the Unknowns:

$\dot{A},\ \frac{2Z}{R}\dot{B},\ \dot{C},\ \dot{C}_o,\ \frac{4Z^2}{R^2}\dot{C}_1,\ \dot{K},\ \frac{8Z^3}{R^3}\ddot{D},\ \frac{4Z}{R^2}\ddot{E},\ \frac{16Z^4}{R^4}\ddot{F},\ \ddot{G},\ u_1,\ u_2$

To determine these 12 unknowns, there are 12 equations expressing the conditions of continuity of $\gamma\dot{w}$, $\gamma\dot{w}'$, $\gamma\dot{v}$, n_ϕ, s, m_ϕ at each of the boundaries $x=u_1$, $x=u_2$.

Let $f]_u$ be the jump of the function f at x=u, we have:

From (2.2.2.2.23) and (2.2.2.2.38):

$$\gamma\dot{w}']_{u_1} = 0 \rightarrow \dot{C}_o = -\frac{2Z}{R}\dot{B}$$

From (2.2.2.2.23) and (2.2.2.2.38):

$$\gamma\dot{w}]_{u_1} = 0 \rightarrow \dot{C} = \dot{Q} + \frac{4Z^2}{R^2}\dot{C}_1$$

From (2.2.2.2.22) and (2.2.2.2.37):

$$\gamma\dot{v}]_{u_1} = 0 \rightarrow -\frac{2Z}{R}u_1\left(\frac{2Z^2}{R^2}\dot{C}_1 + \frac{1}{3}\dot{C}_o u_1\right) = \frac{\dot{A} + \frac{2Z}{R}\dot{B}}{2Z/R} + \frac{2Z}{R}u_1\left(\dot{C} - \frac{Z}{R}\dot{B}u_1\right)$$

From (2.2.2.38) and (2.2.2.2.43):

$$\gamma\dot{w}']_{u_2} = 0 \rightarrow -\frac{2Z}{R}\dot{B} = \dot{K}$$

From (2.2.2.2.38) and (2.2.2.2.43):

$$\gamma\dot{w}]_{u_2} = 0 \rightarrow \dot{C} = -\dot{K}$$

From (2.2.2.2.37) and (2.2.2.2.42):

$$\gamma\dot{v}]_{u_2} = 0 \rightarrow \frac{\dot{A} + \frac{2Z}{R}\dot{B}}{2Z/R} + \frac{2Z}{R} u_2(\dot{C} - \frac{Z}{R}\dot{B}u_2) = \frac{2Z}{R} u_2 \dot{K} (u_2-1)$$

There are 6 equations for 8 unknowns: $\dot{A}$, $\frac{2Z}{R}\dot{B}$, $\dot{C}$, $\dot{C}_o$, $\frac{4Z^2}{R^2}\dot{C}_1$, $\dot{K}$, u_1, u_2. Therefore we can express 6 of them in terms of the other 2 which are $\dot{C}$ and u_1, and we obtain:

$$\dot{K} = -\dot{C} \tag{A1.1}$$

$$\frac{2Z}{R}\dot{B} = \dot{C} \tag{A1.2}$$

$$\dot{C}_o = -\dot{C} \tag{A1.3}$$

$$\frac{4Z^2}{R^2}\dot{C}_1 = \dot{C} - \dot{Q} \tag{A1.4}$$

$$\dot{A} = -\dot{C}(1 + \frac{2Z^2}{R^2} u_2^2) \tag{A1.5}$$

$$u_2^2 = \frac{3\dot{C}-\dot{Q}}{\dot{C}} u_1 - \frac{5}{3} u_1^2 \tag{A1.6}$$

So far we have used the 6 conditions of continuity of $\gamma\dot{w}'$, $\gamma\dot{w}$ and $\gamma\dot{v}$ at u_1 and u_2 to express $\dot{A}$, $\frac{2Z}{R}\dot{B}$, $\dot{C}_o$, $\frac{4Z^2}{R^2}\dot{C}_1$, $\dot{K}$, in terms of $\dot{C}$, u_1 and u_2 and obtain a relation between u_1, u_2 and $\dot{C}$. Now we will go on to determine the 4 remaining unknowns $\frac{8Z^3}{R^3}\ddot{D}$, $\frac{4Z^2}{R^2}\ddot{E}$, $\frac{16Z^4}{R^4}\ddot{F}$, $\ddot{G}$ in terms of $\dot{C}$ and u_1, u_2 by using the 4 conditions of continuity of xs and hxm_ϕ at u_1 and u_2.

Differentiating (A1.5) with respect to τ, we have first:

$$\ddot{A} = -\ddot{C} - \frac{2Z^2}{R^2} (\ddot{C}u_2^2 + 2\dot{C}_2\dot{u}_2u_2) \tag{A1.7}$$

From $xs]_{u_1} = 0$, using (2.2.2.2.21) and (2.2.2.2.40) with $x = u_1$, having $\ddot{A}$, $\frac{2Z}{R}\ddot{B}$, $\ddot{C}_o$, $\frac{4Z^2}{R^2}\ddot{C}_1$ by differentiating with respect to τ (A1.5), (A1.2), (A1.3), (A1.4) respectively

and substituting the results obtained into (2.2.2.2.21) and (2.2.2.2.40), we arrive at:

$$\frac{\frac{4Z^2}{R^2}\ddot{E}-\ddot{Q}}{4Z^2/R^2} - \frac{\frac{8Z^3}{R^3}\ddot{D} - \ddot{C} - \frac{2Z^2}{R^2}(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2)}{4Z^2/R^2}\,u_1 + \frac{1}{2}\,(\ddot{Q} - \frac{4Z^2}{R^2}\ddot{E})\;u_1{}^2$$

$$+ \frac{1}{6}\,(\frac{8Z^3}{R^3}\,\ddot{D} - \ddot{C})\;u_1{}^3 = 0 \tag{A1.8}$$

From $xs]_{u_2} = 0$, using (2.2.2.2.45) and (2.2.2.2.40) with $x = u_2$, and making the necessary substitution, we obtain:

$$h - \ddot{G} + \frac{\frac{4Z^2}{R^2}\ddot{E}-\ddot{Q}}{4Z^2/R^2} - \frac{\frac{8Z^3}{R^3}\ddot{D}-\ddot{C}-\frac{2Z^2}{R^2}(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2)}{4Z^2/R^2}\,u_2$$

$$+ \frac{1}{2}(\ddot{Q}-\frac{4Z^2}{R^2}\ddot{E})\;u_2^2 + \frac{1}{6}(\frac{8Z^3}{R^3}\,\ddot{D} - \ddot{C})\;u_2^3 = 0 \tag{A1.9}$$

From $hxm_\phi]_{u_1} = 0$, using (2.2.2.2.24) and (2.2.2.2.41) with $x = u_1$, having $\ddot{A}$, $\frac{2Z}{R}\ddot{B}$, $\ddot{C}_o$, $\frac{4Z^2}{R^2}\ddot{C}_1$ by differentiating with respect to τ (A1.5), (A1.2), (A1.3), (A1.4) respectively and substituting the results obtained into (2.2.2.2.24) and (2.2.2.2.41), we arrive at:

$$\frac{\frac{16Z^4}{R^4}\ddot{F} + \frac{8Z^3}{R^3}\ddot{D}-\ddot{C}}{16Z^4/R^4} + \frac{\frac{4Z^2}{R^2}\ddot{E}-\ddot{Q}}{4Z^2/R^2}u_1 - \frac{\frac{8Z^3}{R^3}\ddot{D}-\ddot{C} - \frac{2Z^2}{R^2}(\ddot{C}u_2^2 + 2\dot{C}\dot{u}_2u_2)}{8Z^2/R^2}\,u_1^2$$

$$+ \frac{1}{6}\,(\ddot{Q} - \frac{4Z^2}{R^2}\ddot{E})\;u_1^3 + \frac{1}{24}(\frac{8Z^3}{R^3}\ddot{D}-\ddot{C})\;u_1^4 = 0 \tag{A1.10}$$

From $hxm_\phi]_{u_2} = 0$, using (2.2.2.2.46) and (2.2.2.2.41) with $x = u_2$, and making the necessary substitution, we obtain:

$$\frac{1}{12}\ddot{C} + \ddot{G} - \frac{1}{6}\ddot{Q} + \frac{\frac{16Z^4}{R^4}\ddot{F} + \frac{8Z^3}{R^3}\ddot{D}-\ddot{C}}{16Z^4/R^4} + \left[h - \ddot{G} + \frac{\frac{4Z^2}{Z^2}\ddot{E}-\ddot{Q}}{4Z^2/R^2}\right]u_2$$

$$- \frac{\frac{8Z^3}{R^3}\ddot{D}-\ddot{C} - \frac{2Z^2}{R}(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2)}{8Z^2/R^2}\, u_2^2$$

$$+ \frac{1}{6}(\ddot{Q} - \frac{4Z^2}{R^2}\ddot{E})u_2^3 + \frac{1}{24}(\frac{8Z^3}{R^3}\ddot{D}-\ddot{C})u_2^4 = 0 \tag{A1.11}$$

Let:

$$\frac{4Z^2}{R^2}\ddot{E}-\ddot{Q} = \ddot{E}_1 \tag{A1.12}$$

$$\frac{8Z^3}{R^3}\ddot{D}-\ddot{C} = \ddot{D}_1 \tag{A1.13}$$

$$\frac{16Z^4}{R^4}\ddot{F}+\frac{8Z^3}{R}\ddot{D}-\ddot{C} = \ddot{F}_1 \tag{A1.14}$$

Then equations (A1.8)-(A1.11) above become respectively:

$$\frac{\ddot{E}_1}{\frac{4Z^2}{R^2}} - \frac{\ddot{D}_1 - \frac{2Z^2}{R^2}(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2)}{4Z^2/R^2}\, u_1 - \frac{1}{2}\ddot{E}_1u_1^2 + \frac{1}{6}\ddot{D}_1u_1^3 = 0$$

$$h-\ddot{G} + \frac{\ddot{E}_1}{\frac{4Z^2}{R^2}} - \frac{\ddot{D}_1 - \frac{2Z^2}{R^2}(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2)}{4Z^2/R^2}\, u_2 - \frac{1}{2}\ddot{E}u_2^2 + \frac{1}{6}\ddot{D}_1u_2^3 = 0$$

$$\frac{\ddot{F}_1}{\frac{16Z^4}{R^4}} + \frac{\ddot{E}_1}{4Z^2/R^2}\, u_1 - \frac{\ddot{D}_1-\frac{2Z^2}{R^2}(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2)}{8Z^2/R^2}u_1^2 - \frac{1}{8}\ddot{E}_1u_1^3 + \frac{1}{24}\ddot{D}_1u_1^4 = 0$$

2Z

$$\frac{\ddot{C}+12\ddot{G}-2\ddot{Q}}{12}+\frac{\ddot{F}_1}{\frac{16Z^4}{R^4}}+(h-\ddot{G}+\frac{\ddot{E}_1}{\frac{4Z^2}{R^2}})u_2-\frac{\ddot{D}_1-\frac{2Z^2}{R}(\ddot{C}u_2^2+\dot{C}\dot{u}_2u_2)}{8Z^2/R^2}u_2^2-\frac{1}{6}\ddot{E}_1u_2^3+\frac{1}{24}\ddot{D}_1u_2^4=0$$

Elimination of $\ddot{D}_1$ from (A1.15) and (A1.17) yields:

$$\frac{\ddot{F}_1}{4Z^2/R^2} = -\frac{1}{2} u_1 \ddot{E}_1 \tag{A1.19}$$

Substituting this expression of $\ddot{F}_1/[4Z^2/R^2]$ into (A1.18) and multiplying both sides of the resulting equation by -2, we obtain:

$$u_2^2\ddot{D}_1 - (2u_2-u_1)\ddot{E}_1 - \frac{8Z^2}{R^2}(1-u_2)\ddot{G} = \frac{2Z^2}{R^2}[(\ddot{C}u_2^2 + 2\dot{C}\dot{u}_2u_2)u_2 + 4h]u_2 + \frac{2}{3}\frac{Z^2}{R^2}(\ddot{C}-2\ddot{Q}) \tag{A1.20}$$

Elimination of $\ddot{G}$ from (A1.20) and (A1.16) yields:

$$u_2(2-u_2)\ddot{D}_1 - (2-u_1)\ddot{E}_1 = \frac{2Z^2}{R^2}(\ddot{C}u_2^2 + 2\dot{C}\dot{u}_2^2u_2)(2-u_2)u_2 + \frac{2}{3}\frac{Z^2}{R^2}(\ddot{C} - 2\ddot{Q} + 12h) \tag{A1.21}$$

Solving for $\ddot{D}_1$ and $\ddot{E}_1$ from (A1.15) and (A1.21) above, we obtain:

$$\ddot{D}_1 = \frac{2Z^2}{R^2}(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2) + \frac{2}{3}\frac{Z^2}{R^2}\frac{\ddot{C}-2\ddot{Q}+12h}{(u_2-u_1)[2-(u_1+u_2)]} \tag{A1.22}$$

$$\ddot{E}_1 = \frac{2}{3}\frac{Z^2}{R^2}\frac{\ddot{C}-2\ddot{Q}+12h}{(u_2-u_1)[2-(u_1+u_2)]}u_1 \tag{A1.23}$$

From (A1.19), we have:

$$\frac{\ddot{F}_1}{4Z^2/R^2} = -\frac{1}{3}\frac{Z^2}{R^2}\frac{\ddot{C}-2\ddot{Q}+12h}{(u_2-u_1)[2-(u_1+u_2)]}u_1^2 \tag{A1.24}$$

and from (A1.16), we have:

$$\frac{4Z^2}{R^2}\ddot{G} = \frac{2Z^2}{R^2}(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2)u_2 + \frac{4Z^2}{R^2}h + \ddot{E}_1 - u_2\ddot{D}_1$$

With $\ddot{D}_1$ and $\ddot{E}_1$ from (A1.22) and (A1.23), we have:

$$\frac{4Z^2}{R^2}\ddot{G} = \frac{4Z^2}{R^2}h - \frac{2}{3}\frac{Z^2}{R^2}\frac{\ddot{C}-2\ddot{Q}+12h}{2-(u_1+u_2)} \tag{A1.25}$$

Thus far, we have expressed all the unknowns in terms of $\dot{C}$, u_1 and u_2. To determine these 3 unknowns we have the relation (A1.6) and the 2 conditions of continuity of n_ϕ at $x = u_1$ and $x = u_2$.

Thus, from (2.2.2.2.39) and using $\ddot{E}_1$, $\ddot{D}_1$, $\ddot{F}_1$ defined in (A1.12)-(A1.14), and with $\frac{2Z}{R}\ddot{B}$ from (A1.1.2), we have, from $xn_\phi]_{u_1} = 0$:

$$\frac{\ddot{D}_1 - \frac{2Z^2}{R^2}(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2)}{8Z^3/R^3} + \frac{\ddot{E}_1}{2Z/R}u_1 - \frac{\ddot{D}_1}{4Z/R}u_1^2$$

$$+ \frac{1}{3}\frac{Z}{R}(3\ddot{C}-\ddot{Q}-\ddot{E}_1)u_1^3 + \frac{1}{12}\frac{Z}{R}(\ddot{D}_1-5\ddot{C})u_1^4 = 0$$

With $\ddot{D}_1$ and $\ddot{E}_1$ from (A1.22) and (A1.23) and grouping similar terms, we have:

$$\frac{\ddot{C} - 2\ddot{Q} + 12h}{(u_2-u_1)[2-(u_1+u_2)]}\left[\frac{1}{12Z/R} + \frac{1}{6}\frac{Z}{R}u_1^2 - \frac{2}{9}\frac{Z^2}{R^2}u_1^3 + \frac{1}{18}\frac{Z^3}{R^3}u_1^4\right]$$

$$+ \frac{Z}{R}(\ddot{C}u_2^2 + 2\dot{C}\dot{u}_2u_2)\left[-\frac{1}{2}u_1^2 + \frac{1}{6}\frac{Z^2}{R^2}u_1^4\right]$$

$$+ \frac{1}{3}\frac{Z}{R}(3\ddot{C} - \ddot{Q})u_1^3 - \frac{5}{12}\frac{Z}{R}\ddot{C}u_1^4 = 0$$

Using the shallow shell approximation, we have:

$$\ddot{C} - 2\ddot{Q} + 12h = \frac{Z^2}{R^2}(u_2-u_1)\ [2 - (u_1+u_2)]\ [6(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2)u_1^2 - 4(3\ddot{C}-\ddot{Q})u_1^3 + 5\ddot{C}u_1^4] \qquad \text{(A1.26)}$$

Then, we obtain:

$$\frac{d}{d\tau}\ \{\ \dot{C}[1 + 0(\frac{Z^2}{R^2})]\ \ = 2\ddot{Q} - 12h$$

or, approximately:

$$\ddot{C} = 2(\ddot{Q}-6h) \qquad \text{(A1.27a)}$$

With $\ddot{Q}$ defined in (2.2.2.2.35), we have:

$$\ddot{C} = 2(p-p_s) \qquad \text{(A1.27b)}$$

Then:

$$\dot{C} = 2[p_o(1-e^{-\tau}) - p_s\tau] \qquad \text{(A1.28)}$$

Similarly, from $xn_\phi]_{u_2} = 0$, we have, after a first reduction:

$$\frac{\ddot{D}_1 - \frac{2Z^2}{R^2}(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2)}{8Z^3/R^3} + \frac{\ddot{E}_1}{2Z/R}u_2 - \frac{\ddot{D}_1}{4Z/R}u_2^2$$

$$+ \frac{1}{3}\ \frac{Z}{R}(3\ddot{C}-\ddot{Q}-\ddot{E}_1)u_2^3 + \frac{1}{12}\ \frac{Z}{R}(\ddot{D}_1-5\ddot{C})u_2^4 = 0$$

Then with $\ddot{D}_1$ and $\ddot{E}_1$ from (A1.22) and (A1.23), we have:

$$\frac{1}{12Z/R}\ \frac{\ddot{C}-2\ddot{Q}+12h}{(u_2-u_1)[2-(u_1+u_2)]} + \frac{1}{3}\ \frac{Z}{R}\ \frac{\ddot{C}-2\ddot{Q}+12h}{(u_2-u_1)[2-(u_1+u_2)]}\ u_1u_2$$

$$- \frac{Z}{R}[\frac{1}{2}(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2) + \frac{1}{6}\ \frac{\ddot{C}-2\ddot{Q}+12h}{(u_2-u_1)[2-(u_1+u_2)]}\ u_2^2$$

$$+ \frac{1}{3}\ \frac{Z}{R}\ [3\ddot{C}-\ddot{Q} - \frac{2}{3}\ \frac{Z^2}{R^2}\ \frac{\ddot{C}-2\ddot{Q}+12h}{(u_2-u_1)[2-(u_1+u_2)]}\ u_1]\ u_2^3$$

$$+ \frac{1}{12}\ \frac{Z}{R}[\frac{2Z^2}{R^2}(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2) + \frac{2}{3}\ \frac{Z^2}{R^2}\ \frac{\ddot{C}-2\ddot{Q}+12h}{(u_2-u_1)[2-(u_1+u_2)]} - 5\ddot{C}]u_2^4 = 0$$

Using the shallow shell approximation, we obtain:

$$\ddot{C}-2\ddot{Q}+12h = \frac{Z^2}{R^2}(u_2-u_1)\ [2-(u_1+u_2)]\ [6(\ddot{C}u_2^2+2C\dot{u}_2u_2)u_2^2$$

$$-4(3\ddot{C}-\ddot{Q})u_2^3+5\ddot{C}u_2^4] \qquad \text{(A1.29)}$$

By comparing (A1.29) with (A1.26), we obtain a relation between $\dot{C}$, u_1 and u_2 which is:

$$(u_2-u_1)\{6(\ddot{C}u_2^2+2C\dot{u}_2u_2)(u_1+u_2) - 4(3\ddot{C}-\ddot{Q})(u_2^2+u_1u_2+u_1^2)$$

$$+5\ddot{C}(u_1^2+u_2^2)(u_1+u_2)\} = 0 \qquad \text{(A1.30)}$$

where $\ddot{C}$ and $\dot{C}$ have been determined in (A1.27b) and (A1.28) Another relation between $\dot{C}$, u_1, u_2 has been found in (A1.6). Then (A1.30) and (A1.6) are the system of equations to determine u_1 and u_2, $\dot{C}$ and $\ddot{C}$ being known.

From (A1.30) if we take:

$$u_1 = u_2 = u$$

then from (A1.6) we will find:

$$u = \frac{15}{16}\ \frac{p_o(1-e^{-\tau}) - p_s\tau - 1.2h\tau}{p_o(1-e^{-\tau}) - p_s\tau}$$

But we know that this solution is not kinematically admissable, therefore, we must have

$$6(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2)(u_1+u_2) - 4(3\ddot{C}-\ddot{Q})(u_1^2+u_1u_2+u_2^2)$$

$$+5\ddot{C}(u_1^2+u_2^2)(u_1+u_2) = 0 \qquad \text{(A1.31)}$$

and (A1.31) constitutes with (A1.6) the system of equations to determine u_1 and u_2, $\dot{C}$, $\ddot{C}$ and $\ddot{Q}$ being known.

APPENDIX A2

MEDIUM PRESSURE

$6h < p_o - p_s \leq \lambda h$

Determination of the Unknowns:

$\dot{A}$, $\frac{2Z}{R}\dot{B}$, $\dot{C}$, $\dot{C}_o$, $\frac{4Z^2}{R^2}\dot{C}_1$, $\dot{K}$, $\frac{8Z^3}{R^3}\dot{D}$, $\frac{4Z^2}{R^2}\ddot{E}$, $\frac{16Z^4}{R^4}\dot{F}$, $\ddot{G}$, u_o, u_1, u_2

As has been mentioned, to determine these 13 unknowns we have 14 conditions of continuity. If $f]_u$ designate the jump of the function f at $x = u$, then these conditions are:

At $x = u_o$

$$\gamma\dot{w}]_{u_o} = 0 \qquad \gamma\dot{v}]_{u_o} = 0$$

At $x = u_1$ and $x = u_2$

$$m_\phi]_{u_i} = 0 \qquad n_\phi]_{u_i} = 0 \qquad s]_{u_i} = 0$$

$$\gamma\dot{w}']_{u_i} = 0 \qquad \gamma\dot{w}]_{u_i} = 0 \qquad \gamma\dot{v}]_{u_i} = 0$$

$(i = 1,2)$

From (2.2.2.3.4) and (2.2.2.3.10), we have:

$$\gamma\,\dot{w}]_{u_o} = 0 \rightarrow \frac{4Z^2}{R^2}\dot{C}_1 + \dot{C}_o u_o = 0 \qquad \text{(A2.1)}$$

From (2.2.2.3.12), we have:

$$\gamma\dot{v}]_{u_o} = 0 \rightarrow \frac{2Z^2}{R^2}\dot{C}u_o + \frac{1}{3}\dot{C}_o u_o^2 - \frac{1}{u_o}\int_0^\tau \left(\frac{2Z^2}{R^2}\ddot{C}_1 u_o^2 + \frac{1}{3}\ddot{C}_o u_o^3\right)d\tau = 0 \qquad \text{(A2.2)}$$

Integrating by parts we have:

$$\int_0^\tau \left(\frac{2Z^2}{R^2}\ddot{C}_1 u_o^2 + \frac{1}{3}\ddot{C}_o u_o^3\right) d\tau = \frac{2Z^2}{R^2}\dot{C}_1 u_o^2 + \frac{1}{3}\dot{C} u_o^3$$

$$- \int_0^\tau \left(\frac{4Z^2}{R^2}\dot{C}_1 + \dot{C}_o u_o\right) u_o \dot{u}_o d\tau$$

Or with (A2.1) above:

$$\int_0^\tau \left(\frac{2Z^2}{R^2}\ddot{C}_1 u_o^2 + \frac{1}{3}\ddot{C}_o u_o^3\right) d\tau = \frac{2Z^2}{R^2}\dot{C} u_o^2 + \frac{1}{3}\dot{C} u_o^3 \qquad \text{(A2.3)}$$

With this result, (A2.2) becomes:

$$\frac{2Z^2}{R^2}\dot{C} u_o + \frac{1}{3}\dot{C} u_o^2 - \left(\frac{2Z^2}{R^2}\dot{C} u_o + \frac{1}{3}\dot{C} u_o^2\right) \equiv 0$$

Thus, the condition of continuity of $\gamma\dot{v}$ is a consequence of that of $\gamma\dot{w}$ and from these 2 conditions, we have only one equation: (A2.1).

From (2.2.2.3.9) and (2.2.2.2.38), we have:

$$\gamma\dot{w}']_{u_1} = 0 \rightarrow \dot{C}_o = -\frac{2Z}{R}\dot{B} \qquad \text{(A2.4)}$$

From (2.2.2.3.9) and (2.2.2.2.38) and with (A2.3) above:

$$\gamma\dot{w}]_{u_1} = 0 \rightarrow \dot{Q} + \frac{4Z^2}{R^2}\dot{C}_1 = \dot{C} \qquad \text{(A2.5)}$$

From (2.2.2.3.12) and (2.2.2.2.37) and with (A2.3):

$$\gamma\dot{v}]_{u_1} = 0 \rightarrow -\frac{2Z}{R}\left[\frac{2Z^2}{R^2}\dot{C}_1 u_1 + \frac{1}{3}\dot{C}_o u_1^2 - \frac{1}{u_1}\left(\frac{2Z^2}{R^2}\dot{C}_1 + \frac{1}{3}\dot{C}_o u_o\right)u_o^2\right] = \frac{\dot{A} + \frac{2Z}{R}\dot{B}}{2Z/R} + \frac{2Z}{R} u_1 \left(\dot{C} - \frac{Z}{R}\dot{B}u_1\right) \qquad (A2.6)$$

From (2.2.2.2.38) and (2.2.2.2.43) we have:

$$\gamma\dot{w}']_{u_2} = 0 \rightarrow -\frac{2Z}{R}\dot{B} = \dot{K} \qquad (A2.7)$$

and with this result:

$$\gamma\dot{w}]_{u_2} = 0 \rightarrow \dot{C} = -\dot{K} \qquad (A2.8)$$

From (2.2.2.2.37) and (2.2.2.2.42), we have:

$$\gamma\dot{v}]_{u_2} = 0 \rightarrow \frac{\dot{A} + \frac{2Z}{R}\dot{B}}{2Z/R} + \frac{2Z}{R} u_2 \left(\dot{C} - \frac{Z}{R}\dot{B}u_2\right)$$

$$= \frac{2Z}{R}\dot{K}\, u_2(u_2-1) \qquad (A2.9)$$

As before, we will first express $\dot{A}$, $\frac{2Z}{R}\dot{B}$, $\dot{C}_o$, $\frac{4Z^2}{R^2}\dot{C}_1$, $\dot{K}$ in terms of $\dot{C}$, u_o, u_1, u_2 using equations (A2.4), (A2.5) and (A2.7)-(A2.9).

From (A2.8):

$$\dot{K} = -\dot{C} \qquad (A2.10)$$

From (A2.7) and using the result above:

$$\frac{2Z}{R}\dot{B} = \dot{C} \qquad (A2.11)$$

From (A2.4) and using the result above:

$$\dot{C}_o = -\dot{C} \tag{A2.12}$$

From (A2.5)

$$\frac{4Z^2}{R^2}\dot{C}_1 = \dot{C} - \dot{Q} \cdot \tag{A2.13}$$

From (A2.9) and using (A2.10) and (A2.11) above, we have:

$$\frac{\dot{A} + \frac{2Z}{R}\dot{B}}{2Z/R} = -\frac{Z}{R}\dot{C}\,u_2^2 \tag{A2.14a}$$

or, with (A2.11):

$$\dot{A} = -\dot{C}(1 + \frac{2Z^2}{R^2}u_2^2) \tag{A2.14b}$$

and we are left with two unused relations (A2.1) and (A2.6) and which will be used later in the determination of $\dot{C}$, u_1, u_2, u_o.

From (2.2.2.3.7) and (2.2.2.2.40), we have:

$$xs]_{u_1} = 0 \rightarrow \frac{2Z^2}{R^2}\ddot{C}_1(u_1^2-u_o^2) + \frac{1}{3}\ddot{C}_o(u_1^3-u_o^3) =$$

$$= \frac{\frac{4Z^2}{R^2}\ddot{E}-\ddot{Q}}{4Z^2/R^2} - \frac{\ddot{A} + \frac{8Z^3}{R^3}\ddot{D}}{4Z^2/R^2}u_1 + \frac{1}{2}(\ddot{C} - \frac{4Z^2}{R^2}\ddot{E})u_1^2 +$$

$$+ \frac{1}{6}(\frac{8Z^3}{R^3}\ddot{D} - \frac{6Z}{R}\ddot{B})u_1^3$$

Having $\frac{2Z}{R}\ddot{B}$, $\ddot{C}_o$, $\frac{4Z^2}{R^2}\ddot{C}_1$ and $\ddot{A}$ by differentiating (A2.11)-(A2.13) and (A2.14b) respectively, we have:

$$\begin{gathered}\frac{\frac{4Z^2}{R^2}\ddot{E}-\ddot{Q}}{4Z^2/R^2} - \frac{\frac{8Z^3}{R^3}\ddot{D} - \ddot{C} - \frac{2Z^2}{R^2}(\ddot{C}u_2^2 + 2\dot{C}\dot{u}_2u_2)}{4Z^2/R^2}u_1 + \\ \frac{1}{2}(\ddot{Q} - \frac{4Z^2}{R^2}\ddot{E})u_1^2 + \frac{1}{6}(\frac{8Z^3}{R^3}\ddot{D}-\ddot{C})u_1^3 + \frac{\ddot{C}-\ddot{Q}}{2}u_o^2 - \frac{1}{3}\ddot{C}u_o^3 = 0\end{gathered} \tag{A2.15}$$

From (2.2.2.2.40) and (2.2.2.2.45), we have:

$$xs]_{u_2} = 0 \rightarrow \frac{\frac{4Z^2}{R^2}\ddot{E}-\ddot{Q}}{4Z^2/R^2} - \frac{\ddot{A} + \frac{8Z^3}{R^3}\ddot{D}}{4Z^2/R^2} u_2 +$$

$$+ \frac{1}{2}\left(\ddot{C} - \frac{4Z^2}{R^2}\ddot{E}\right) u_2^2 + \frac{1}{6}\left(\frac{8Z^3}{R^3}\ddot{D} - \frac{6Z}{R}\ddot{B}\right) u_2^3 =$$

$$= \ddot{G} - h - \frac{\ddot{K} + \ddot{Q}}{2} u_2^2 + \frac{1}{3} K u_2^3$$

Substituting again $\ddot{A}$, $\frac{2Z}{R}\ddot{B}$, $\ddot{K}$ by their expressions in terms of $\dot{C}$ and u_2, we have:

$$\frac{\frac{4Z^2}{R^2}\ddot{E}-\ddot{Q}}{4Z^2/R^2} + h - \ddot{G} - \frac{\frac{8Z^3}{R^3}\ddot{D} - \ddot{C} - \frac{2Z^2}{R^2}\left(\ddot{C}u_2^2 + 2\dot{C}\dot{u}_2u_2\right)}{4Z^2/R^2} u_2$$

$$+ \frac{1}{2}\left(\ddot{Q} - \frac{4Z^2}{R^2}\ddot{E}\right)u_2^2 + \frac{1}{6}\left(\frac{8Z^3}{R^3}\ddot{D} - \ddot{C}\right)u_2^3 = 0 \qquad \text{(A2.16)}$$

From (2.2.2.3.13) and (2.2.2.2.41), we have:

$$hxm_{\phi}]_{u_1} = 0 \rightarrow hu_1 + \frac{2}{3}\frac{Z^2}{R^2}\ddot{C}_1\left(u_1^3 - 3u_o^2u_1 + 2u_o^2\right) +$$

$$+ \frac{1}{12}\ddot{C}_o\left(u_1^4 - 4u_o^3u_1 + 3u_o^4\right) =$$

$$= \frac{\frac{16Z^4}{R^4}\ddot{F} + \frac{8Z^3}{R^3}\ddot{D} - \frac{2Z}{R}\ddot{B}}{16Z^4/R^4} + \left[h + \frac{\frac{4Z^2}{R^2}\ddot{E}-\ddot{Q}}{4Z^2/R^2}\right]u_1 - \frac{\ddot{A} + \frac{8Z^3}{R^3}\ddot{D}}{8Z^2/R^2} u_1^2$$

$$+ \frac{1}{6}\left(\ddot{C} - \frac{4Z^2}{R^2}\ddot{E}\right)u_1^3 + \frac{1}{24}\left(\frac{8Z^3}{R^3}\ddot{D} - \frac{6Z}{R}\ddot{B}\right)u_1^4$$

Having $\frac{2Z}{R}\ddot{B}, \ddot{C}_o$, $\frac{4Z^2}{R^2}\ddot{C}_1$ and $\ddot{A}$ by differentiating (A2.11)-(A2.13) and (A2.14b), respectively, we have:

$$\frac{\frac{16Z^4}{R^4}\ddot{F}+\frac{8Z^3}{R^3}\ddot{D}-\ddot{C}}{16Z^4/R^4}+\frac{\frac{4Z^2}{R^2}\ddot{E}-\ddot{Q}}{4Z^2/R^2}u_1-\frac{\frac{8Z^3}{R^3}\ddot{D}-\ddot{C}-\frac{2Z^2}{R^2}(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2)}{8Z^2/R^2}u_1^2$$

$$+\frac{1}{6}(\ddot{Q}-\frac{4Z^2}{R^2}\ddot{E})\,u_1^3+\frac{1}{24}(\frac{8Z^3}{R^3}\ddot{D}-\ddot{C})\,u_1^4+\frac{\ddot{C}-\ddot{Q}}{2}u_o^2u_1-\frac{\ddot{C}-\ddot{Q}}{3}u_o^3$$

$$-\frac{1}{3}\ddot{C}u_o^3u_1+\frac{1}{4}\ddot{C}u_o^4=0$$

(A2.17)

From (2.2.2.2.41) and (2.2.2.2.46) and making the substitution of $\frac{2Z}{R}\ddot{B}$, $\ddot{A}$ and $\ddot{K}$ in terms of $\dot{C}$ and u_2 wherever possible, as before, we have:

$$hxm_\phi]_{u_2}=0\rightarrow$$

(A2.18)

$$\frac{\frac{16Z^4}{R^2}\ddot{F}+\frac{8Z^3}{R^3}\ddot{D}-\ddot{C}}{16Z^4/R^4}+\frac{1}{12}\ddot{C}-\frac{1}{6}\ddot{Q}+\ddot{G}+[h+\frac{\frac{4Z^2}{R^2}\ddot{E}-\ddot{Q}}{4Z^2/R^2}-\ddot{G}]u_2$$

$$-\frac{\frac{8Z^3}{R^3}\ddot{D}-\ddot{C}-\frac{2Z^2}{R^2}(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2)}{8Z^2/R^2}u_2^2+\frac{1}{6}(\ddot{Q}-\frac{4Z^2}{R^2}\ddot{E})\,u_2^3+\frac{1}{24}(\frac{8Z^3}{R^3}\ddot{D}-\ddot{C})\,u_2^4=0$$

Using $\ddot{E}_1$, $\ddot{D}_1$, $\ddot{F}_1$ defined in (A1.12)-(A1.14) respectively, equations (A2.15)-(A2.18) can be written as:

From (A2.15):

$$\frac{\ddot{E}_1}{4Z^2/R^2}-\frac{\ddot{D}_1-\frac{2Z^2}{R^2}(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2)}{4Z^2/R^2}u_1$$

$$-\frac{\ddot{E}_1}{2}u_1^2+\frac{\ddot{D}_1}{6}u_1^3+\frac{\ddot{C}-\ddot{Q}}{2}u_o^2\;\frac{1}{3}\ddot{C}u_o^3=0$$

Multiplying both sides by $4Z^2/R^2$ and grouping similar terms, we have:

$$-\left(1-\frac{2}{3}\frac{Z^2}{R^2}u_1^2\right)u_1\ddot{D}_1+\left(1-\frac{2Z^2}{R^2}u_1^2\right)\ddot{E}_1=$$

$$-\frac{2Z^2}{R^2}[(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2)u_1+(\ddot{C}-\ddot{Q})u_o^2-\frac{2}{3}\ddot{C}u_o^3] \qquad \text{(A2.19)}$$

From (A2.16):

$$\frac{\ddot{E}_1}{4Z^2/R^2}+h-\ddot{G}-\frac{\ddot{D}_1-\frac{2Z^2}{R^2}(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2)}{4Z^2/R^2}u_2-$$

$$-\frac{\ddot{E}_1}{2}u_2^2+\frac{\ddot{D}_1}{6}u_2^3=0$$

Multiplying both sides by $4Z^2/R^2$ and grouping similar terms, we have:

$$-\left(1-\frac{2}{3}\frac{Z^2}{R^2}u_2^2\right)u_2\ddot{D}_1+\left(1-\frac{2Z^2}{R^2}u_2^2\right)\ddot{E}_1-\frac{4Z^2}{R^2}\ddot{G}=$$

$$=-\frac{2Z^2}{R^2}[(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2)u\ +2h] \qquad \text{(A2.20)}$$

From (A2.17):

$$\frac{\ddot{F}_1}{16Z^4/R^4}+\frac{\ddot{E}_1u_1}{4Z^2/R^2}-\frac{\ddot{D}_1-\frac{2Z^2}{R^2}(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2)}{8Z^2/R^2}u_1^2-\frac{\ddot{E}_1}{6}u_1^3+$$

$$+\frac{\ddot{D}_1}{24}u_1^4+\frac{\ddot{C}-\ddot{Q}}{2}u_o^2u_1-\frac{\ddot{C}-\ddot{Q}}{3}u_o^3-\frac{1}{3}\ddot{C}u_o^3u_1+\frac{1}{4}\ddot{C}u_o^4=0$$

Multiplying both sides by $4Z^2/R^2$ and grouping similar terms, we have:

$$-\left(\frac{1}{2}-\frac{1}{6}\frac{Z^2}{R^2}u_1^2\right)u_1^2\ddot{D}_1+\left(1-\frac{2}{3}\frac{Z^2}{R^2}u_1^2\right)u_1\ddot{E}_1+\frac{\ddot{F}_1}{4Z^2/R^2}=$$

$$-\frac{Z^2}{R^2}[(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2)u_1^2+2(\ddot{C}-\ddot{Q})u_o^2u_1-$$

$$-\frac{4}{3}(\ddot{C}-\ddot{Q})u_o^3-\frac{4}{3}\ddot{C}u_o^3u_1+\ddot{C}u_o^4] \qquad \text{(A2.21)}$$

From (A2.18):

$$\frac{\ddot{F}_1}{16Z^4/R^4} + \frac{1}{12}\ddot{C} - \frac{1}{6}\ddot{Q} + \ddot{G} + \left(h - \frac{\ddot{E}_1}{4Z^2/R^2} - \ddot{G}\right) u_2 -$$

$$- \frac{\ddot{D}_1 - \frac{2Z^2}{R^2}(\ddot{C}u_2^2 + 2\dot{C}\dot{u}_2u_2)}{8Z^2/R^2} u_2^2 - \frac{\ddot{E}_1}{6} u_2^3 + \frac{\ddot{D}_1}{24} u_2^4 = 0$$

Multiplying both sides by $4Z^2/R^2$ and grouping similar terms, we have:

$$- \left(\frac{1}{2} - \frac{1}{6}\frac{Z^2}{R^2} u_2^2\right)u_2^2\ddot{D}_1 + \left(1 - \frac{2}{3}\frac{Z^2}{R^2} u_2^2\right)u_2\ddot{E}_1 +$$

$$+ \frac{4Z^2}{R^2}(1-u_2)\ddot{G} + \frac{\ddot{F}_1}{4Z^2/R^2}$$

$$= - \frac{Z^2}{R^2}\left[(\ddot{C}u_2^2 + 2\dot{C}\dot{u}_2u_2)u_2^2 + 4hu_2 + \frac{\ddot{C}-2\ddot{Q}}{3} \right] \tag{A2.22}$$

With the shallow shell approximation:

$$\frac{4Z^2}{R^2} u_1 \leq \frac{4Z^2}{R^2} u_2 \leq \frac{4Z^2}{R^2} << 1,$$

equations (A2.19)-(A2.22) become:

$$u_1\ddot{D}_1-\ddot{E}_1 = \frac{2Z^2}{R^2}\left[(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2)u_1 + (\ddot{C}-\ddot{Q})u_o^2 - \frac{2}{3}\ddot{C}u_o^3\right] \tag{A2.23}$$

$$u_2\ddot{D}_1-\ddot{E}_1 + \frac{4Z^2}{R^2}\ddot{G} = \frac{2Z^2}{R^2}\left[(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2)u_2 + 2h\right] \tag{A2.24}$$

$$\frac{1}{2}u_1^2\ddot{D}_1-u_1\ddot{E}_1 - \frac{\ddot{F}_1}{4Z^2/R^2} = \frac{Z^2}{R^2}\left[(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2)u_1^2 + 2(\ddot{C}-\ddot{Q})u_o^2u_1 - \right.$$

$$\left. - \frac{4}{3}(\ddot{C}-\ddot{Q})u_o^3 - \frac{4}{3}\ddot{C}u_o^3u_1 + \ddot{C}u_o^4\right] \tag{A2.25}$$

$$\frac{1}{2}u_2^2\ddot{D}_1 - u_2\ddot{E}_1 - \frac{4Z^2}{R^2}(1-u_2)\ddot{G} - \frac{\ddot{F}_1}{4Z^2/R^2} =$$

$$\frac{Z^2}{R^2}\left[\ (\ddot{C}u_2^2 + 2\dot{C}\dot{u}_2u_2)u_1^2 + 4hu_2 + \frac{\ddot{C}-2\ddot{Q}}{3}\ \right] \qquad \text{(A2.26)}$$

With the equations (A2.23)-(A2.26), we can solve for $\ddot{D}_1$, $\ddot{E}_1$, $\ddot{F}_1$ and $\ddot{G}$ in terms of $\dot{C}$ and u_o, u_1, u_2. We proceed by elimination: multiplying (A2.23) by u_1 and (A2.25) by -2 and adding the resulting equations, we obtain:

$$u_1\ddot{E}_1 + \frac{2\ddot{F}_1}{4Z^2/R^2} = \frac{2Z^2}{R^2}\left[-\ (\ddot{C}-\ddot{Q})u_o^2u_1 + \frac{4}{3}(\ddot{C}-\ddot{Q})u_o^3 + \frac{2}{3}\ddot{C}u_o^2u_1 - \ddot{C}u_o^4\right]$$

From this, we have:

$$\frac{\ddot{F}_1}{4Z^2/R^2} = -\ \frac{1}{2}\ u_1\ddot{E}_1 +$$

$$+ \frac{Z^2}{R^2}\left[\ -\ (\ddot{C}-\ddot{Q})u_o^2u_1 + \frac{4}{3}\ (\ddot{C}-\ddot{Q})u_o^3 + \frac{2}{3}\ \ddot{C}u_o^2u_1 - \ddot{C}u_o^4\ \right]$$

$$\text{(A2.27)}$$

Substituting this expression of $F_1/(4Z^2/R^2)$ into (A2.26), we have:

$$\frac{1}{2}u_2^2\ddot{D}_1 - u_2\ddot{E}_1 - \frac{4Z^2}{R^2}(1-u_2)\ddot{G} + \frac{1}{2}u_1\ddot{E}_1$$

$$= \frac{Z^2}{R^2}\left[(\ddot{C}u_2^2 + 2\dot{C}\dot{u}_2u_2)u_2^2 + 4hu_2 + \frac{\ddot{C}-2\ddot{Q}}{3} - (\ddot{C}-\ddot{Q})u_o^2u_1 + \right.$$

$$\left. + \frac{4}{3}(\ddot{C}-\ddot{Q})u_o^3 + \frac{2}{3}\ddot{C}u_o^2u_1 - \ddot{C}u_o^4\right]$$

Multiplying both sides by 2 and grouping similar terms, we obtain:

$$u_2^2\ddot{D}_1 - (2u_2-u_1)\ddot{E}_1 - \frac{8Z^2}{R^2}(1-u_2)\ddot{G}$$

$$= \frac{2Z^2}{R^2}\,[(\ddot{C}u_2^2 + 2\dot{C}\dot{u}_2u_2)u_2^2 + 4hu_2 + \frac{\ddot{C}-2\ddot{Q}}{3} - (\ddot{C}-\ddot{Q})u_o^2u_1 +$$

$$+ \frac{4}{3}(\ddot{C}-\ddot{Q})u_o^3 + \frac{2}{3}\ddot{C}u_o^2u_1 - \ddot{C}u_o^4] \qquad (A2.28)$$

Multiplying (A2.24) by $2(1-u_2)$ and adding the resulting equation to (A2.28) above to eliminate $\ddot{G}$, we have:

$$u_2(2-u_2)\ddot{D}_1 - (2-u_1)\ddot{E}_1 = \frac{2Z^2}{R^2}[(\ddot{C}u_2^2 + 2\dot{C}\dot{u}_2u_2)(2-u_2)u_2 + 4h$$

$$+ \frac{\ddot{C}-2\ddot{Q}}{3} - (\ddot{C}-\ddot{Q})u_o^2u_1 + \frac{4}{3}(\ddot{C}-\ddot{Q})u_o^3 + \frac{2}{3}\ddot{C}u_o^2u_1 - \ddot{C}u_o^4]$$

(A2.29)

With (A2.23) and (A2.29) above, we can solve for $\ddot{D}_1$ and $\ddot{E}_1$. To eliminate $\ddot{E}_1$, we multiply (A2.23) by $-(2-u_1)$ and add the result to (A2.29). The result, after some operations, is:

$$\ddot{D}_1 = \frac{2Z^2}{R^2}(\ddot{C}u_2^2+2\dot{C}\dot{u}_2u_2) + \frac{2Z^2/R^2}{(u_2-u_1)[2-(u_1+u_2)]} \times$$

$$\times \left[\frac{\ddot{C}-2\ddot{Q}+12h}{3} - 2(\ddot{C}-\ddot{Q})u_o^2 + \frac{4}{3}(2\ddot{C}-\ddot{Q})u_o^3 - \ddot{C}u_o^4\right]$$

(A2.30)

From (A2.23), we have:

$$\ddot{E}_1 = \frac{\frac{2Z^2}{R^2}\,u_1}{(u_2-u_1)[2-(u_1+u_2)]}\,\left[\frac{\ddot{C}-2\ddot{Q}+12h}{3} - 2(\ddot{C}-\ddot{Q})u_o^2 +\right.$$

$$\left. + \frac{4}{3}(2\ddot{C}-\ddot{Q})u_o^3 - \ddot{C}u_o^4\right] - \frac{2Z^2}{R^2}\,\left[(\ddot{C}-\ddot{Q})u_o^2 - \frac{2}{3}\ddot{C}u_o^3\right]$$

(A2.31)

With $\ddot{E}_1$ given by (A2.31), expression (A2.27) for $\ddot{F}_1$ becomes:

$$\frac{\ddot{F}_1}{4Z^2/R^2} = - \frac{\frac{Z^2}{R^2}\, u_1^2}{(u_2-u_1)\,[2-(u_1+u_2)]}\, [\frac{\ddot{C}-2\ddot{Q}+12h}{3} - 2(\ddot{C}\ -\ddot{Q})u_o^2 +$$

$$+ \frac{4}{3}(2\ddot{C}-\ddot{Q})u_o^3 - \ddot{C}u_o^4] + \frac{Z^2}{R^2}\, [\frac{4}{3}(\ddot{C}-\ddot{Q})u_o^3 - \ddot{C}u_o^4]$$

(A2.32)

To obtain $\ddot{G}$, we can use equation (A2.24), with $\ddot{E}_1$ and $\ddot{D}_1$ from (A2.31) and (A2.30) respectively:

$$\frac{4Z^2}{R^2}\ddot{G} = \frac{4Z^2}{R^2}h - \frac{2Z^2/R^2}{[2-(u_1+u_2)]}\, [\frac{\ddot{C}-2\ddot{Q}+12h}{3} - 2(\ddot{C}-\ddot{Q})u_o^2 +$$

$$+ \frac{4}{3}(2\ddot{C}-\ddot{Q})u_o^3 - \ddot{C}u_o^4] - \frac{2Z^2}{R^2}\, [(\ddot{C}-\ddot{Q})u_o^2 - \frac{2}{3}\ddot{C}u_o^2]$$

(A2.33a)

Then:

$$\ddot{G} = h - \frac{1}{2}\, [(\ddot{C}-\ddot{Q}) - \frac{2}{3}\ddot{C}u_o]u_o^2 - \frac{1}{2[2-(u_1+u_2)]}\, [\frac{\ddot{C}-2\ddot{Q}+12h}{3} -$$

$$- 2(\ddot{C}-\ddot{Q})u_o^2 + \frac{4}{3}(2\ddot{C}-\ddot{Q})u_o^3 - \ddot{C}u_o^4] \qquad \text{(A2.33b)}$$

We observe that when $u_o = 0$, expressions (A2.30)-(A2.32) and (A2.33a) reduce to expressions (A1.22)-(A1.25) of the medium pressure case.

So far we have expressed all the unknowns in terms of $\dot{C}$, u_o, u_1, u_2. To determine these 4 remaining unknowns, we have relations (A2.1), (A2.6) and the 2 conditions of continuity

of n_ϕ at $x = u_1$ and $x = u_2$. From (2.2.2.2.39) and having $\ddot{A}$ and $\frac{2Z}{R}B$ by differentiating (A2.14b) and (A2.11) respectively and using $\ddot{D}_1$ and $\ddot{E}_1$ defined in (A1.12),(A1.14), we obtain:

$$xn_\phi]_{u_1} = 0 \rightarrow \frac{\ddot{D}_1 - \frac{2Z^2}{R^2}(\ddot{C}u_2^2 + 2\dot{C}\dot{u}_2u_2)}{8Z^3/R^3} -$$

$$- \frac{\ddot{D}_1u_1^2}{4Z\;/R}\left(1 - \frac{1}{3}\frac{Z^2}{R^2}u_1^2\right) + \frac{\ddot{E}_1u_1}{2Z/R}\left(1 - \frac{2}{3}\frac{Z^2}{R^2}u_1^2\right) +$$

$$+ \frac{Z}{R}\left(\frac{3\ddot{C}-\ddot{Q}}{3}u_1^3 - \frac{5\ddot{C}}{12}u_1^4\right) = 0$$

With $\ddot{D}_1$ and $\ddot{E}_1$ from (A2.30) and (A2.31) respectively, and using the shallow shell approximation, this equation becomes:

$$\frac{1}{\frac{4Z}{R}(u_2-u_1)[2-(u_1+u_2)]}\left[\frac{\ddot{C}-2\ddot{Q}+12h}{3} - 2(\ddot{C}-\ddot{Q})u_o^2 + \frac{4}{3}(2\ddot{C}-\ddot{Q})u_o^3 - \ddot{C}u_o^4\right]$$

$$= \frac{Z}{R}\left[\frac{1}{2}(\ddot{C}u_2^2 + 2\dot{C}\dot{u}_2u_2)u_1^2 - \frac{3\ddot{C}-\ddot{Q}}{3}u_1^3 + \frac{5\ddot{C}}{12}u_1^4 + (\ddot{C}-\ddot{Q})u_o^2u_1\right.$$

$$\left. - \frac{2}{3}\ddot{C}u_o^3u_1\right] \qquad \text{(A2.34)}$$

Similarly, at $x = u_2$, we have:

$$xn_\phi]_{u_2} = 0 \rightarrow \frac{1}{\frac{4Z}{R}(u_2-u_1)[2-(u_1+u_2)]}\left[\frac{\ddot{C}-2\ddot{Q}+12h}{3} -\right.$$

$$\left. - 2(\ddot{C}-\ddot{Q})u_o^2 + \frac{4}{3}(2\ddot{C}-\ddot{Q})u_o^3 - \ddot{C}u_o^4\right] = \frac{Z}{R}\left[\frac{1}{2}(\ddot{C}u_2^2 + 2\dot{C}\dot{u}_2u_2)u_2^2 -\right.$$

$$\left. - \frac{3\ddot{C}-\ddot{Q}}{3}u_2^3 + \frac{5\ddot{C}}{12}u_2^4 + (\ddot{C}-\ddot{Q})u_o^2u_2 - \frac{2}{3}\ddot{C}u_o^3u_2\right] \qquad \text{(A2.35)}$$

Comparing this equality with (A2.34), we obtain:

$$\frac{1}{2}(\ddot{C}u_2^2 + 2\dot{C}\dot{u}_2u_2)(u_2^2-u_1^2) - \frac{3\ddot{C}-\ddot{Q}}{3}(u_2^3-u_1^3) + \frac{5\ddot{C}}{12}(u_2^4-u_1^4)$$

$$+ (\ddot{C}-\ddot{Q})u_o^2(u_2-u_1) - \frac{2\ddot{C}}{3}u_o^3(u_2-u_1) = 0$$

or:

$$(u_2-u_1)\left\{\frac{1}{2}(\ddot{C}u_2^2 + 2\dot{C}\dot{u}_2u_2)(u_1+u_2) - \frac{3\ddot{C}-\ddot{Q}}{3}(u_1^2+u_1u_2+u_2^2)\right.$$

$$\left. + \frac{5\ddot{C}}{12}(u_1+u_2)(u_1^2+u_2^2) + (\ddot{C}-\ddot{Q})u_o^2 - \frac{2\ddot{C}}{3}u_o^3\right\} = 0$$

From the result of the medium pressure case, we have found that $u_1 \neq u_2$ and therefore $u_2 - u_1 \neq 0$. Thus a relation between u_o, u_1, u_2 and $\dot{C}$ is, after some reductions:

$$6(\ddot{C}u_2^2 + 2\dot{C}\dot{u}_2u_2)(u_1+u_2) - 4(3\ddot{C}-\ddot{Q})(u_1^2+u_1u_2+u_2^2)$$

$$+ 5\ddot{C}(u_1+u_2)(u_1^2+u_2^2) + 12(\ddot{C}-\ddot{Q})u_o^2 - 8\ddot{C}u_o^3 = 0 \qquad \text{(A2.36)}$$

Comparing this relation with (A1.31) we see that the former will reduce to the latter when $u_o = 0$.

Also, from (A2.34), we have:

$$\frac{\ddot{C} - 2\ddot{Q} + 12h}{3} = 2(\ddot{C}-\ddot{Q})u_o^2 - \frac{4}{3}(2\ddot{C}-\ddot{Q})u_o^3 + \ddot{C}u_o^4$$

$$+ \frac{4Z^2}{R^2}\left[\frac{1}{2}(\ddot{C}u_2^2 + 2\dot{C}\dot{u}_2u_2)u_1^2 - \frac{3\ddot{C}-\ddot{Q}}{3}u_1^3 + \frac{5\ddot{C}}{12}u_1^4\right.$$

$$\left. + (\ddot{C}-\ddot{Q})u_o^2u_1 - \frac{2}{3}\ddot{C}u_o^3u_1\right] \qquad \text{(A2.37)}$$

or, using the shallow shell approximation:

$$\ddot{C} = 2(\ddot{Q}-6h) + 6(\ddot{C}-\ddot{Q})u_o^2 - 4(2\ddot{C}-\ddot{Q})u_o^3 + 3\ddot{C}u_o^4 \qquad (A2.38)$$

From (A2.1), which is:

$$\frac{4Z^2}{R^2}\dot{C}_1 + \dot{C}_o u_o = 0$$

and with $\frac{4Z^2}{R^2}\dot{C}_1$ from (A2.13), C_o from (A2.12), we have:

$$\dot{C} = \frac{\dot{Q}}{1 - u_o} \qquad (A2.39)$$

from which we derive:

$$\ddot{C} = \frac{(1-u_o)\ddot{Q} + \dot{Q}\dot{u}_o}{(1-u_o)^2} \qquad (A2.40)$$

We can also compute $\ddot{C}$ from (A2.38):

$$\ddot{C} = \frac{2(1-3u_o^2 + 2u_o^3)\ddot{Q} - 12h}{1 - 6u_o^2 + 8u_o^3 - 3u_o^4} \qquad (A2.41a)$$

or:

$$\ddot{C} = \frac{2(1-u_o)^2(1+2u_o)\ddot{Q} - 12h}{(1-u_o)^3(1+3u_o)} \qquad (A2.41b)$$

Comparing (A2.40) with (A2.41b), we obtain, after some reductions:

$$(1-u_o)^2(1+u_o)\ddot{Q} - (1-u_o)(1+3u_o)\dot{Q}\dot{u}_o = 12h \text{ , or}$$

$$\frac{d}{d\tau}[(1-u_o)^2(1+u_o)\dot{Q}] = 12h$$

Integrating, we obtain:

$$(1-u_o)^2(1+u_o)\dot{Q} = 12h\tau + L$$

Since at $\tau = 0$, $\dot{Q} = 0$, we have $L = 0$ and:

$$(1-u_o)^2(1+u_o) = \frac{12h\tau}{\dot{Q}} \tag{A2.42a}$$

With $\dot{Q}$ defined in (2.2.2.2.47), we have:

$$(1-u_o)^2(1+u_o) = \frac{12h\tau}{p_o(1-e^{-\tau}) - p_s\tau + 6h\tau} \tag{A2.42b}$$

Using a formula in the solution of an equation of 3rd degree which has 3 real roots [21], it can be shown that u_o can be expressed in the form:

$$u_o = \frac{1}{3} - \frac{4}{3}\cos\left(\frac{\pi+\phi}{3}\right) \tag{A2.42c}$$

where ϕ is determined by:

$$\phi = \cos^{-1}\left\{\frac{81h\tau}{4[p_o(1-e^{-\tau}) - p_s\tau + 6h\tau]} - 1\right\} \quad \text{for } \tau > 0 \tag{A2.42d}$$

and

$$\phi = \cos^{-1}\left[\frac{81h}{4(p_o-p_s + 6h)} - 1\right] \qquad \text{for } \tau = 0$$

At $\tau = 0$, the right-hand side has the undeterminate form 0/0. Using the L'Hospital's rule, we have, at $\tau = 0$:

$$(1-u_{oo})^2(1+u_{oo}) = \frac{12h}{p_o-p_s + 6h} \tag{A2.43}$$

From this, we see that if $p_o-p_s = 6h$, then $u_{oo} = 0$ and the shell collapses only in 3 regimes as has been found in the medium pressure case. ($u_{oo} = u_o(0)$)

Thus we have determined $\dot{C}$, $\ddot{C}$ and u_o. To obtain u_1 and u_2, we have 2 relations: (A2.6) and (A2.36).

From (A2.6) and with $\frac{4Z^2}{R^2}\dot{C}_1$, $\dot{C}_o$, $\frac{2Z}{R}\dot{B}$, $\dot{A}$ from (A2.13), (A2.12), (A2.11) and (A2.14b), respectively, we have, after some reductions:

$$3(3\dot{C}-\dot{Q})u_1^2 - 5\dot{C}u_1^3 - 3\dot{C}u_2^2u_1 - 3(\dot{C}-\dot{Q})u_o^2 + 2\dot{C}u_o^3 = 0 \tag{A2.44}$$

With $\dot{C}$ from (A2.39), we have:

$$3\dot{C} - \dot{Q} = \dot{Q}\,\frac{2+u_o}{1-u_o}$$

$$\dot{C} - \dot{Q} = \dot{Q}\,\frac{u_o}{1-u_o}$$

With these results, (A2.44) becomes, after simplification by $\frac{\dot{Q}}{1-u_o}$:

$$3(2+u_o)u_1^2 - 5u_1^3 - 3u_2^2u_1 - u_o^3 = 0 \qquad \text{or:}$$

$$u_2^2 = \frac{3(2+u_o)u_1^2 - 5u_1^3 - u_o^3}{3u_1} \tag{A2.45}$$

We observe also that when $u_o = 0$, (A2.44) will reduce formally to (A1.6) of the medium pressure case.

Thus relation (A2.6) becomes (A2.45) above and relation (A2.36) remains the same.

$\ddot{C}$, $\ddot{Q}$, u_o being known, we can determine then u_1, u_2 from (A2.45) and (A2.36)

With $\dot{C}$ and $\ddot{C}$ known in terms of u_o, u_1, u_2, we can express all other unknowns in terms of u_o, u_1, u_2, and hence express the velocities and stresses in terms of u_o, u_1, u_2.

APPENDIX A3

SAMPLE OUTPUT FOR MEDIUM PRESSURE RANGE
$(1.2h \leq p_o - p_s \leq 6h)$

The symbols used in this output are:

ZETA = $(Z/R)/h$

DPI = $(p_o - p_s)/h$

RP = p_o/p_s

Q0 = g_o (initial value of g)

TSTP = τ_f

TAUF = τ_1

RMAX = maximum value of $\dfrac{1+n_\theta}{h(Z/R)}$

RREST, REST1, REST2 are the approximations with which the solutions of the equations to obtain Q0, TSTP and TAUF have been made (the closer to 0 the better)

IR, IF1, IF2 are indications of possible error in the evaluations of Q0, TSTP and TAUF. The value 0 means no error.

M is the number of points in the numerical integration of equation (2.2.2.2.74)

TAU = τ, the independent variable in equation (2.2.2.2.74)

Q = g

U1 = u_1

U2 = u_2

DQ = $1/\dot{g}$

P(U1) and P(U2) are values of P(x)/h where P(x) is defined in (2.2.2.2.92) for $x = u_1$ and $x = u_2$ respectively.

$DD1 = u_1 - S/2$

$DD2 = u_2 - S/2$

where S is the sum of the two root of the equation

$$P(x) = 0$$

AD1 is condition (2.2.2.2.89) where the function in the right-hand side has been transferred to the left-hand side.

AD2 is condition (2.2.2.2.95) for $p-p_s > 0$ and condition (2.2.2.1.19) for $p-p_s < 0$ (for $p-p_s < 0$, it has been proved that n_θ is admissible).

It can be seen from this output that for $\tau \rightarrow \tau_1$, $g \rightarrow \infty$.

```
                         CONDITIONS  FOR  ADMISSIBILITY

                         1-AC1  POSITIVE

                         2-AD2  POSITIVE

                         3-RMAX  SMALLER  THAN  1./H*Z


ZETA     CP1     RP      CO      TSTP    TAUF    RMAX    WREST   REST1   REST2  IR  IF1 IF2  N
12.00    6.00    1.11    1.45    0.21    0.17    6.52    -0.00   0.00    0.00   0   0   0   20


 TAU      C          U1      U2       D0          P(U1)     P(U2)     AD1        AD2       AC1       AC2
0.000   0.149E 01   0.513   0.766   0.153E 01    0.00      1.45      0.35       0.61      0.09      0.10
0.015   0.151E 01   0.497   0.750   0.153E 01    0.12      1.36      0.35       0.61      0.09      0.17
0.028   0.152E 01   0.483   0.733   0.120E 01    0.24      1.31      0.37       0.62      0.09      0.24
0.039   0.153E 01   0.468   0.717   0.978E 00    0.36      1.29      0.39       0.64      0.10      0.31
0.049   0.154E 01   0.454   0.702   0.815E 00    0.47      1.31      0.42       0.66      0.10      0.39
0.058   0.156E 01   0.441   0.686   0.698E 00    0.58      1.33      0.45       0.70      0.10      0.46
0.065   0.157E 01   0.427   0.671   0.608E 00    0.68      1.37      0.52       0.76      0.10      0.54
0.072   0.158E 01   0.415   0.656   0.536E 00    0.78      1.42      0.60       0.84      0.10      0.61
0.079   0.160E 01   0.402   0.642   0.479E 00    0.87      1.47      0.72       0.96      0.11      0.69
0.083   0.161E 01   0.390   0.628   0.431E 00    0.95      1.53      0.89       1.13      0.11      0.75
0.092   0.163E 01   0.368   0.601   0.359E 00    1.09      1.64      1.53       1.76      0.11      0.88
0.100   0.166E 01   0.347   0.576   0.303E 00    1.21      1.74      3.81       4.04      0.11      1.02
0.106   0.168E 01   0.327   0.551   0.259E 00    1.30      1.84    -22.07     -21.84      0.11      6.41
0.112   0.171E 01   0.308   0.526   0.223E 00    1.38      1.93     -3.60      -3.39      0.12      6.41
0.117   0.173E 01   0.290   0.503   0.195E 00    1.44      2.01     -2.23      -2.02      0.12      6.41
0.121   0.176E 01   0.273   0.481   0.170E 00    1.49      2.07     -1.75      -1.54      0.12      6.41
0.125   0.178E 01   0.257   0.459   0.150E 00    1.52      2.12     -1.52      -1.31      0.12      6.41
0.129   0.181E 01   0.242   0.438   0.133E 00    1.53      2.16     -1.34      -1.19      0.12      6.41
0.132   0.184E 01   0.228   0.418   0.118E 00    1.54      2.19     -1.31      -1.11      0.12      6.41
0.134   0.186E 01   0.215   0.399   0.106E 00    1.54      2.20     -1.25      -1.07      0.12      6.41
0.139   0.191E 01   0.191   0.365   0.859E-01    1.51      2.20     -1.21      -1.03      0.11      6.41
0.143   0.196E 01   0.170   0.334   0.703E-01    1.45      2.17     -1.14      -1.02      0.11      6.41
0.146   0.201E 01   0.152   0.305   0.585E-01    1.39      2.11     -1.13      -1.03      0.11      6.41
0.148   0.206E 01   0.135   0.279   0.489E-01    1.32      2.04     -1.19      -1.04      0.11      6.41
0.150   0.211E 01   0.121   0.256   0.411E-01    1.24      1.95     -1.20      -1.06      0.10      6.41
0.152   0.216E 01   0.109   0.236   0.349E-01    1.16      1.87     -1.21      -1.08      0.10      6.41
0.154   0.221E 01   0.098   0.217   0.298E-01    1.08      1.78     -1.22      -1.10      0.10      6.41
0.155   0.226E 01   0.088   0.200   0.256E-01    1.01      1.70     -1.23      -1.11      0.09      6.41
0.156   0.232E 01   0.080   0.185   0.221E-01    0.94      1.61     -1.24      -1.13      0.09      6.41
0.157   0.237E 01   0.073   0.172   0.192E-01    0.83      1.52     -1.24      -1.15      0.09      6.41
0.159   0.247E 01   0.061   0.150   0.148E-01    0.75      1.37     -1.25      -1.17      0.08      6.41
0.160   0.257E 01   0.051   0.131   0.116E-01    0.66      1.23     -1.27      -1.19      0.08      6.41
0.161   0.267E 01   0.044   0.116   0.927E-02    0.58      1.11     -1.29      -1.21      0.07      6.41
0.162   0.277E 01   0.037   0.104   0.751E-02    0.51      1.01     -1.30      -1.23      0.07      6.41
0.163   0.287E 01   0.032   0.093   0.617E-02    0.45      0.92     -1.30      -1.24      0.06      6.41
0.163   0.297E 01   0.028   0.084   0.513E-02    0.40      0.84     -1.31      -1.26      0.06      6.41
0.164   0.307E 01   0.025   0.077   0.432E-02    0.36      0.77     -1.32      -1.27      0.06      6.41
0.164   0.317E 01   0.022   0.070   0.367E-02    0.32      0.71     -1.32      -1.27      0.06      6.41
0.164   0.328E 01   0.020   0.065   0.315E-02    0.29      0.65     -1.33      -1.28      0.05      6.41
0.165   0.338E 01   0.018   0.060   0.272E-02    0.26      0.61     -1.33      -1.29      0.05      6.41
0.165   0.358E 01   0.015   0.052   0.209E-02    0.22      0.53     -1.34      -1.30      0.05      6.41
0.165   0.378E 01   0.012   0.046   0.164E-02    0.18      0.47     -1.34      -1.31      0.05      6.41
0.166   0.398E 01   0.010   0.041   0.132E-02    0.16      0.41     -1.34      -1.31      0.04      6.41
0.166   0.418E 01   0.009   0.037   0.108E-02    0.14      0.37     -1.35      -1.32      0.04      6.41
0.166   0.439E 01   0.008   0.033   0.897E-03    0.12      0.34     -1.35      -1.32      0.04      6.41
0.166   0.459E 01   0.007   0.030   0.755E-03    0.10      0.31     -1.35      -1.33      0.04      6.41
0.166   0.479E 01   0.006   0.028   0.644E-03    0.09      0.28     -1.35      -1.33      0.04      6.41
0.167   0.499E 01   0.005   0.026   0.554E-03    0.08      0.26     -1.35      -1.33      0.04      6.41
0.167   0.520E 01   0.005   0.024   0.482E-03    0.07      0.24     -1.35      -1.34      0.04      6.41
0.167   0.540E 01   0.004   0.022   0.422E-03    0.07      0.23     -1.36      -1.34      0.03      6.41
0.167   0.580E 01   0.003   0.020   0.331E-03    0.06      0.20     -1.36      -1.34      0.03      6.41
0.167   0.621E 01   0.003   0.018   0.266E-03    0.05      0.18     -1.36      -1.34      0.03      6.41
0.167   0.661E 01   0.002   0.016   0.218E-03    0.04      0.16     -1.36      -1.35      0.03      6.41
0.167   0.701E 01   0.002   0.015   0.182E-03    0.03      0.14     -1.36      -1.35      0.03      6.41
0.167   0.742E 01   0.002   0.013   0.154E-03    0.03      0.13     -1.36      -1.35      0.03      6.41
0.167   0.782E 01   0.002   0.012   0.132E-03    0.03      0.12     -1.36      -1.35      0.03      6.41
0.167   0.823E 01   0.001   0.012   0.114E-03    0.02      0.11     -1.36      -1.35      0.03      6.41
0.167   0.863E 01   0.001   0.011   0.996E-04    0.02      0.10     -1.36      -1.35      0.03      6.41
0.167   0.904E 01   0.001   0.010   0.877E-04    0.02      0.10     -1.36      -1.35      0.03      6.41
0.167   0.944E 01   0.001   0.009   0.778E-04    0.02      0.09     -1.36      -1.35      0.03      6.41
0.168   0.102E 02   0.001   0.008   0.624E-04    0.01      0.08     -1.36      -1.36      0.03      6.41
0.168   0.111E 02   0.001   0.008   0.511E-04    0.01      0.07     -1.36      -1.36      0.03      6.41
0.168   0.119E 02   0.001   0.007   0.426E-04    0.01      0.07     -1.36      -1.36      0.02      6.41
0.168   0.127E 02   0.001   0.006   0.360E-04    0.01      0.06     -1.36      -1.36      0.02      6.41
0.168   0.135E 02   0.000   0.006   0.308E-04    0.01      0.06     -1.36      -1.36      0.02      6.41
0.168   0.143E 02   0.000   0.006   0.267E-04    0.01      0.05     -1.36      -1.36      0.02      6.41
0.168   0.151E 02   0.000   0.005   0.233E-04    0.01      0.05     -1.36      -1.36      0.02      6.41
0.168   0.159E 02   0.000   0.005   0.206E-04    0.01      0.05     -1.36      -1.36      0.02      6.41
0.168   0.167E 02   0.000   0.005   0.183E-04    0.00      0.04     -1.36      -1.36      0.02      6.41
0.168   0.175E 02   0.000   0.004   0.163E-04    0.00      0.04     -1.36      -1.36      0.02      6.41
0.168   0.191E 02   0.000   0.004   0.133E-04    0.00      0.04     -1.37      -1.36      0.02      6.41
0.168   0.208E 02   0.000   0.004   0.110E-04    0.00      0.03     -1.37      -1.36      0.02      6.41
0.168   0.224E 02   0.000   0.003   0.926E-05    0.00      0.03     -1.37      -1.36      0.02      6.41
0.168   0.240E 02   0.000   0.003   0.790E-05    0.00      0.03     -1.37      -1.36      0.02      6.41
0.168   0.256E 02   0.000   0.003   0.682E-05    0.00      0.03     -1.37      -1.36      0.02      6.41
0.168   0.272E 02   0.000   0.003   0.595E-05    0.00      0.02     -1.37      -1.36      0.02      6.41
0.168   0.288E 02   0.000   0.002   0.523E-05    0.00      0.02     -1.37      -1.36      0.02      6.41
0.168   0.305E 02   0.000   0.002   0.464E-05    0.00      0.02     -1.37      -1.36      0.02      6.41
0.168   0.321E 02   0.000   0.002   0.414E-05    0.00      0.02     -1.37      -1.36      0.02      6.41
0.168   0.337E 02   0.000   0.002   0.372E-05    0.00      0.02     -1.37      -1.36      0.02      6.41
0.168   0.369E 02   0.000   0.002   0.305E-05    0.00      0.02     -1.37      -1.36      0.02      6.41
0.168   0.402E 02   0.000   0.002   0.254E-05    0.00      0.02     -1.37      -1.36      0.02      6.41
0.168   0.434E 02   0.000   0.002   0.215E-05    0.00      0.01     -1.37      -1.36      0.02      6.41
0.168   0.466E 02   0.000   0.001   0.184E-05    0.00      0.01     -1.37      -1.36      0.02      6.41
0.168   0.499E 02   0.000   0.001   0.160E-05    0.00      0.01     -1.37      -1.36      0.02      6.41
0.168   0.531E 02   0.000   0.001   0.140E-05    0.00      0.01     -1.37      -1.36      0.02      6.41
0.168   0.563E 02   0.000   0.001   0.124E-05    0.00      0.01     -1.37      -1.36      0.02      6.41
0.168   0.596E 02   0.000   0.001   0.110E-05    0.00      0.01     -1.37      -1.36      0.02      6.41
0.168   0.628E 02   0.000   0.001   0.983E-06    0.00      0.01     -1.37      -1.36      0.02      6.41
0.168   0.660E 02   0.000   0.001   0.885E-06    0.00      0.01     -1.37      -1.36      0.02      6.41
0.168   0.725E 02   0.000   0.001   0.724E-06    0.00      0.01     -1.37      -1.37      0.02      6.41
0.168   0.790E 02   0.000   0.001   0.609E-06    0.00      0.01     -1.37      -1.37      0.02      6.41
0.168   0.854E 02   0.000   0.001   0.518E-06    0.00      0.01     -1.37      -1.37      0.02      6.41
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0.168  0.919E 02  0.000  0.001  0.445E-06  0.00  0.01  -1.37  -1.37  0.02  6.41
0.168  0.984E 02  0.000  0.001  0.387E-06  0.00  0.01  -1.37  -1.37  0.02  6.41
0.168  0.105E 03  0.000  0.001  0.339E-06  0.00  0.01  -1.37  -1.37  0.02  6.41
0.168  0.111E 03  0.000  0.001  0.300E-06  0.00  0.01  -1.37  -1.37  0.02  6.41
0.168  0.118E 03  0.000  0.001  0.267E-06  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.124E 03  0.000  0.001  0.239E-06  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.131E 03  0.000  0.000  0.216E-06  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.144E 03  0.000  0.000  0.178E-06  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.157E 03  0.000  0.000  0.149E-06  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.170E 03  0.000  0.000  0.127E-06  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.182E 03  0.000  0.000  0.109E-06  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.195E 03  0.000  0.000  0.951E-07  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.208E 03  0.000  0.000  0.835E-07  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.221E 03  0.000  0.000  0.739E-07  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.234E 03  0.000  0.000  0.659E-07  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.247E 03  0.000  0.000  0.591E-07  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.260E 03  0.000  0.000  0.533E-07  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.286E 03  0.000  0.000  0.440E-07  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.312E 03  0.000  0.000  0.369E-07  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.338E 03  0.000  0.000  0.314E-07  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.364E 03  0.000  0.000  0.271E-07  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.389E 03  0.000  0.000  0.236E-07  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.415E 03  0.000  0.000  0.207E-07  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.441E 03  0.000  0.000  0.183E-07  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.467E 03  0.000  0.000  0.164E-07  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.493E 03  0.000  0.000  0.147E-07  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.519E 03  0.000  0.000  0.132E-07  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.571E 03  0.000  0.000  0.109E-07  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.622E 03  0.000  0.000  0.918E-08  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.674E 03  0.000  0.000  0.782E-08  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.726E 03  0.000  0.000  0.674E-08  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.777E 03  0.000  0.000  0.587E-08  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.829E 03  0.000  0.000  0.516E-08  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.881E 03  0.000  0.000  0.457E-08  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.933E 03  0.000  0.000  0.408E-08  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.984E 03  0.000  0.000  0.366E-08  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.104E 04  0.000  0.000  0.330E-08  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.114E 04  0.000  0.000  0.273E-08  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.124E 04  0.000  0.000  0.229E-08  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.135E 04  0.000  0.000  0.195E-08  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.145E 04  0.000  0.000  0.169E-08  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.155E 04  0.000  0.000  0.147E-08  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.166E 04  0.000  0.000  0.129E-08  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.176E 04  0.000  0.000  0.114E-08  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.186E 04  0.000  0.000  0.102E-08  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.197E 04  0.000  0.000  0.913E-09  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.207E 04  0.000  0.000  0.824E-09  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.228E 04  0.000  0.000  0.681E-09  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.248E 04  0.000  0.000  0.572E-09  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.269E 04  0.000  0.000  0.487E-09  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.290E 04  0.000  0.000  0.420E-09  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.311E 04  0.000  0.000  0.366E-09  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.331E 04  0.000  0.000  0.322E-09  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.352E 04  0.000  0.000  0.285E-09  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.373E 04  0.000  0.000  0.254E-09  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.393E 04  0.000  0.000  0.228E-09  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.414E 04  0.000  0.000  0.206E-09  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.455E 04  0.000  0.000  0.170E-09  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.497E 04  0.000  0.000  0.143E-09  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.538E 04  0.000  0.000  0.122E-09  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.580E 04  0.000  0.000  0.105E-09  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.621E 04  0.000  0.000  0.915E-10  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.662E 04  0.000  0.000  0.804E-10  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.704E 04  0.000  0.000  0.712E-10  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.745E 04  0.000  0.000  0.635E-10  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.787E 04  0.000  0.000  0.570E-10  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.828E 04  0.000  0.000  0.514E-10  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.911E 04  0.000  0.000  0.425E-10  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.994E 04  0.000  0.000  0.357E-10  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.108E 05  0.000  0.000  0.304E-10  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.116E 05  0.000  0.000  0.262E-10  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.124E 05  0.000  0.000  0.229E-10  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.132E 05  0.000  0.000  0.201E-10  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.141E 05  0.000  0.000  0.178E-10  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.149E 05  0.000  0.000  0.159E-10  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.157E 05  0.000  0.000  0.142E-10  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.166E 05  0.000  0.000  0.129E-10  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.182E 05  0.000  0.000  0.106E-10  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.199E 05  0.000  0.000  0.893E-11  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.215E 05  0.000  0.000  0.761E-11  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.232E 05  0.000  0.000  0.656E-11  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.248E 05  0.000  0.000  0.572E-11  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.265E 05  0.000  0.000  0.502E-11  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.281E 05  0.000  0.000  0.445E-11  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.298E 05  0.000  0.000  0.397E-11  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.315E 05  0.000  0.000  0.356E-11  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.331E 05  0.000  0.000  0.322E-11  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.364E 05  0.000  0.000  0.266E-11  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.397E 05  0.000  0.000  0.223E-11  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.431E 05  0.000  0.000  0.190E-11  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.464E 05  0.000  0.000  0.164E-11  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.497E 05  0.000  0.000  0.143E-11  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.530E 05  0.000  0.000  0.126E-11  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.563E 05  0.000  0.000  0.111E-11  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.596E 05  0.000  0.000  0.992E-12  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.629E 05  0.000  0.000  0.891E-12  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.662E 05  0.000  0.000  0.804E-12  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.729E 05  0.000  0.000  0.664E-12  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.795E 05  0.000  0.000  0.559E-12  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.861E 05  0.000  0.000  0.476E-12  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.927E 05  0.000  0.000  0.410E-12  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.993E 05  0.000  0.000  0.357E-12  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.106E 06  0.000  0.000  0.314E-12  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.113E 06  0.000  0.000  0.278E-12  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.119E 06  0.000  0.000  0.244E-12  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.126E 06  0.000  0.000  0.223E-12  0.00  0.00  -1.37  -1.37  0.02  6.41
0.168  0.132E 06  0.000  0.000  0.201E-12  0.00  0.00  -1.37  -1.37  0.02  6.41
```

APPENDIX A4

SAMPLE OUTPUT FOR MEDIUM HIGH PRESSURE RANGE

$(6h \leq p_o - p_s \leq \lambda h)$

The symbols used in this output are:

Depth Ratio = depth/thickness

Pressure Difference = $(p_o - p_s)/h$

Pressure Ratio = p_o/p_s

TAUF = τ_f

TAU1 = τ_1

TAUO = τ_o

REST1, REST2, REST3 and IF1, IF2, IF3 are related to the subprogramme subroutine that solves for τ_f, τ_1, τ_o and have the same significations as in Appendix A3.

RPO = $(p_o e^{-\tau_o} - p_s)/h$

RP1 = $(p_o e^{-\tau_1} - p_s)/h$

RPF = $(p_o e^{-\tau_f} - p_s/h$

TAU = τ

U0 = u_o

U1 = u_1

U2 = u_2

AK1 is condition (2.2.2.3.43)

DAM and DBM are coefficients of the x^3 term and x^2 term respectively of the polynomial PM(x) inside the curly brackets of the m'_ϕ expression defined in (2.2.2.3.46).

DPM2 = PM(1) value of PM(x) at x = 1.

DNP1 = u_1 - S/2

DNP2 = u_2 - S/2

where S is the sum of the roots of the equation:

$$P(x) = 0$$

P(x) being the polynomial of 2nd degree inside the curly brackets of the xn_ϕ expression in (2.2.2.3.37b).

PNP1 = $P(u_1)$

PNP2 = $P(u_2)$

DNT1 = $R'(u_2)$

DNT2 = $R'(1)$

where R'(x) is the derivative of R(x), which is the polynomial inside the curly brackets of the n_θ expression in (2.2.2.3.40a)

PNT1 = $R(u_2)$

PNT2 = $R(1)$

Other symbols in the second stage of motion have the same signification as in Appendix A3.

```
                              FIRST  STAGE  OF  MOTION

                         DEPTH  RATIO = 4     PRESSURE  DIFFERENCE =12.00     PRESSURE RATIO = 1.17

      TAUF    REST1      IF1     TAU1   REST2      IF2    TAUC   REST3      IF3      U1     REST        IR
     0.325    0.000        0    0.299   0.000        0   0.154  -0.000        0   0.594    0.000         0

                  RPO=    0.300      RP1=  -9.527      RPF= -10.769

 TAU    U1     U     U2      AK1     DAY     DRM    DRM2    DNP1    DNP2    PNP1    PNP2    DNT1    DNT2    PNT1    PNT2
0.00   0.28  0.59  0.87     -0.3    74.8    -6.9    65.4     0.4     0.7     0.0     3.1    25.9    42.9     0.4     4.9
0.02   0.26  0.59  0.86     -0.3    66.0    -3.9    60.9     0.4     0.7     0.2     2.9    22.9    38.9     0.5     4.8
0.03   0.24  0.58  0.85     -0.3    57.7    -1.0    56.3     0.4     0.7     0.3     2.8    20.2    34.7     0.6     4.6
0.05   0.22  0.57  0.85     -0.3    49.7     1.9    52.0     0.4     0.7     0.4     2.6    17.6    30.9     0.7     4.4
0.06   0.20  0.56  0.84     -0.3    42.1     4.5    47.7     0.4     0.7     0.5     2.4    15.2    27.1     0.8     4.2
0.08   0.17  0.55  0.83     -0.3    34.8     7.3    43.5     0.5     0.7     0.7     2.3    12.9    23.5     0.9     4.0
0.09   0.14  0.54  0.82     -0.3    27.9    10.0    39.5     0.5     0.8     0.8     2.1    10.9    20.0     1.1     3.8
0.11   0.11  0.53  0.81     -0.3    21.1    12.9    35.5     0.5     0.8     0.9     2.0     9.1    16.6     1.2     3.6
0.12   0.08  0.52  0.79     -0.3    14.5    15.9    31.7     0.7     1.0     1.1     2.0     7.4    13.3     1.3     3.4
0.14   0.04  0.51  0.78     -0.3     8.2    19.0    28.1     1.0     1.3     1.3     2.0     6.0    10.1     1.5     3.2
0.15  -0.00  0.49  0.76     -0.3     1.8    22.9    24.6     4.1     4.4     1.5     2.1     4.8     6.9     1.7     3.1

                              SECOND  STAGE  OF  MOTION

        TAU        U1        U2     P(U1)     P(U2)      DD1      DD2      AD1      AD2         Q
      0.154     0.486     0.760     1.772     2.471     4.109    4.394    0.114    1.072      0.156E 01
      0.194     0.414     0.690     2.672     3.192    -0.523   -0.248    0.137    1.232      0.166E 01
      0.220     0.349     0.616     3.345     4.065    -0.455   -0.188    0.152    1.232      0.176E 01
      0.237     0.291     0.543     3.696     4.733    -0.517   -0.265    0.159    1.232      0.186E 01
      0.249     0.243     0.477     3.749     5.106    -0.595   -0.360    0.160    1.232      0.196E 01
      0.257     0.203     0.418     3.623     5.217    -0.669   -0.453    0.158    1.232      0.206E 01
      0.263     0.170     0.367     3.392     5.138    -0.734   -0.537    0.154    1.232      0.216E 01
      0.267     0.143     0.324     3.114     4.940    -0.790   -0.609    0.148    1.232      0.226E 01
      0.270     0.121     0.286     2.824     4.675    -0.837   -0.671    0.143    1.232      0.236E 01
      0.273     0.103     0.255     2.546     4.382    -0.876   -0.724    0.137    1.232      0.246E 01
      0.276     0.078     0.207     2.069     3.822    -0.934   -0.804    0.127    1.232      0.266E 01
      0.278     0.060     0.172     1.687     3.318    -0.975   -0.864    0.119    1.232      0.286E 01
      0.280     0.047     0.145     1.387     2.889    -1.006   -0.908    0.110    1.232      0.306E 01
      0.281     0.039     0.125     1.153     2.534    -1.029   -0.942    0.103    1.232      0.326E 01
      0.282     0.031     0.109     0.970     2.239    -1.046   -0.969    0.097    1.232      0.346E 01
      0.283     0.026     0.096     0.825     1.996    -1.059   -0.990    0.092    1.232      0.366E 01
      0.283     0.022     0.085     0.710     1.793    -1.070   -1.006    0.088    1.232      0.386E 01
      0.284     0.019     0.077     0.616     1.622    -1.078   -1.020    0.084    1.232      0.406E 01
      0.284     0.015     0.070     0.539     1.477    -1.085   -1.032    0.081    1.232      0.426E 01
      0.284     0.014     0.064     0.476     1.353    -1.091   -1.042    0.078    1.232      0.446E 01
      0.285     0.011     0.055     0.378     1.156    -1.100   -1.057    0.074    1.232      0.486E 01
      0.285     0.009     0.047     0.308     1.005    -1.107   -1.068    0.070    1.232      0.526E 01
      0.286     0.007     0.042     0.255     0.886    -1.111   -1.077    0.067    1.232      0.566E 01
      0.286     0.006     0.037     0.215     0.790    -1.115   -1.084    0.065    1.232      0.606E 01
      0.286     0.005     0.034     0.184     0.713    -1.118   -1.090    0.063    1.232      0.646E 01
      0.286     0.004     0.031     0.159     0.648    -1.121   -1.094    0.061    1.232      0.686E 01
      0.286     0.004     0.029     0.139     0.593    -1.123   -1.098    0.060    1.232      0.726E 01
      0.286     0.003     0.026     0.122     0.547    -1.124   -1.102    0.059    1.232      0.766E 01
      0.286     0.003     0.024     0.109     0.507    -1.126   -1.105    0.057    1.232      0.806E 01
      0.286     0.003     0.023     0.096     0.472    -1.127   -1.107    0.056    1.232      0.846E 01
      0.287     0.002     0.020     0.078     0.414    -1.129   -1.111    0.054    1.232      0.926E 01
      0.287     0.002     0.018     0.065     0.369    -1.131   -1.115    0.053    1.232      0.101E 02
      0.287     0.001     0.016     0.054     0.333    -1.132   -1.117    0.052    1.232      0.109E 02
      0.287     0.001     0.015     0.046     0.302    -1.133   -1.119    0.051    1.232      0.117E 02
      0.287     0.001     0.014     0.040     0.277    -1.134   -1.121    0.050    1.232      0.125E 02
      0.287     0.001     0.013     0.035     0.256    -1.134   -1.123    0.049    1.232      0.133E 02
      0.287     0.001     0.012     0.031     0.237    -1.135   -1.124    0.049    1.232      0.141E 02
      0.287     0.001     0.011     0.027     0.221    -1.135   -1.125    0.048    1.232      0.149E 02
      0.287     0.001     0.010     0.024     0.207    -1.136   -1.126    0.048    1.232      0.157E 02
      0.287     0.001     0.010     0.022     0.195    -1.136   -1.127    0.047    1.232      0.165E 02
      0.287     0.000     0.009     0.018     0.174    -1.137   -1.129    0.046    1.232      0.181E 02
      0.287     0.000     0.008     0.015     0.157    -1.137   -1.130    0.046    1.232      0.197E 02
      0.287     0.000     0.007     0.013     0.143    -1.138   -1.131    0.045    1.232      0.213E 02
      0.287     0.000     0.007     0.011     0.132    -1.138   -1.132    0.045    1.232      0.229E 02
      0.287     0.000     0.006     0.009     0.122    -1.139   -1.132    0.045    1.232      0.245E 02
      0.287     0.000     0.006     0.008     0.113    -1.139   -1.133    0.044    1.232      0.261E 02
      0.287     0.000     0.005     0.007     0.106    -1.139   -1.134    0.044    1.232      0.277E 02
      0.287     0.000     0.005     0.006     0.099    -1.139   -1.134    0.044    1.232      0.293E 02
      0.287     0.000     0.005     0.006     0.093    -1.139   -1.134    0.043    1.232      0.309E 02
      0.287     0.000     0.004     0.005     0.088    -1.139   -1.135    0.043    1.232      0.325E 02
      0.287     0.000     0.004     0.004     0.080    -1.139   -1.136    0.043    1.232      0.357E 02
      0.287     0.000     0.004     0.004     0.072    -1.140   -1.136    0.043    1.232      0.389E 02
      0.287     0.000     0.003     0.003     0.066    -1.140   -1.136    0.042    1.232      0.421E 02
      0.287     0.000     0.003     0.003     0.061    -1.140   -1.137    0.042    1.232      0.453E 02
      0.287     0.000     0.003     0.002     0.057    -1.140   -1.137    0.042    1.232      0.485E 02
      0.287     0.000     0.003     0.002     0.053    -1.140   -1.137    0.042    1.232      0.517E 02
      0.287     0.000     0.003     0.002     0.050    -1.140   -1.138    0.042    1.232      0.549E 02
      0.287     0.000     0.002     0.002     0.047    -1.140   -1.138    0.042    1.232      0.581E 02
      0.287     0.000     0.002     0.001     0.044    -1.140   -1.138    0.041    1.232      0.613E 02
      0.287     0.000     0.002     0.001     0.042    -1.140   -1.138    0.041    1.232      0.645E 02
      0.287     0.000     0.002     0.001     0.038    -1.140   -1.139    0.041    1.232      0.709E 02
      0.287     0.000     0.002     0.001     0.035    -1.141   -1.139    0.041    1.232      0.773E 02
      0.287     0.000     0.002     0.001     0.032    -1.141   -1.139    0.041    1.232      0.837E 02
      0.287     0.000     0.002     0.001     0.030    -1.141   -1.139    0.041    1.232      0.901E 02
      0.287     0.000     0.001     0.001     0.028    -1.141   -1.139    0.041    1.232      0.965E 02
      0.287     0.000     0.001     0.000     0.026    -1.141   -1.139    0.041    1.232      0.103E 03
      0.287     0.000     0.001     0.000     0.024    -1.141   -1.140    0.041    1.232      0.109E 03
      0.287     0.000     0.001     0.000     0.023    -1.141   -1.140    0.041    1.232      0.116E 03
      0.287     0.000     0.001     0.000     0.022    -1.141   -1.140    0.041    1.232      0.122E 03
      0.287     0.000     0.001     0.000     0.020    -1.141   -1.140    0.040    1.232      0.128E 03
      0.287     0.000     0.001     0.000     0.019    -1.141   -1.140    0.040    1.232      0.141E 03
      0.287     0.000     0.001     0.000     0.017    -1.141   -1.140    0.040    1.232      0.154E 03
      0.287     0.000     0.001     0.000     0.016    -1.141   -1.140    0.040    1.232      0.167E 03
      0.287     0.000     0.001     0.000     0.015    -1.141   -1.140    0.040    1.232      0.180E 03
      0.287     0.000     0.001     0.000     0.014    -1.141   -1.140    0.040    1.232      0.192E 03
      0.287     0.000     0.001     0.000     0.013    -1.141   -1.140    0.040    1.232      0.205E 03
      0.287     0.000     0.001     0.000     0.012    -1.141   -1.141    0.040    1.232      0.218E 03
      0.287     0.000     0.001     0.000     0.011    -1.141   -1.141    0.040    1.232      0.231E 03
      0.287     0.000     0.001     0.000     0.011    -1.141   -1.141    0.040    1.232      0.244E 03
      0.287     0.000     0.001     0.000     0.010    -1.141   -1.141    0.040    1.232      0.256E 03
```

APPENDIX A5

INSCRIBING AND CIRCUMSCRIBING THE EXACT TRESCA YIELD SURFACE BY THE UNCOUPLED DIAMOND YIELD SURFACE

I. Circumscribing

The uncoupled diamond yield surface $f_D = 1$ (Table 2.1.3.3) is a convex polyhedron. To have this surface circumscribe the exact Tresca yield surface $f_T = 1$, it is necessary to expand the polyhedron until it is exterior to the surface $f_T = 1$. This will be the case if each of the eight planes of $f_D = K$ $(K > 1)$ is exterior, or tangent, to the surface at the point where the normals to the two surfaces are the same. Because of symmetry and since the planes $n_1-n_2 = \pm 1$ and $m_1-m_2 = \pm 1$ of $f_D = 1$ are already exterior to the surface $f_T = 1$, it is sufficient to consider the two planes: $n_1+n_2 = 1$ and $m_1+m_2 = 1$.

Using a procedure analogous to Section 3.7 of Reference [17], we find that the corresponding circumscribing planes are respectively:

$$(P1) \equiv n_1+n_2 = 2, \quad (P2) \equiv m_1+m_2 = 2$$

the points of contact being (1,1,0,0) for (P1) and (0,0,1,1) for (P2).

Therefore if the surface $f_D = 1$ is subjected to an expansion of ratio 2 it will be exterior to the surface $f_T = 1$.

II. Inscribing

Because of the convexity of the yield surfaces and since the surface $f_D = 1$ is a polyhedron, to have it inscribe the exact Tresca yield surface $f_T = 1$, it will be sufficient to require the vertices of $f_D = f$ $(0 < f < 1)$ to lie on or within $f_T = 1$.

Furthermore, since both surfaces are symmetric with respect to moments and direct stresses, only 2 of the 16 vertices need be considered. They are: $P(0,f,0,f)$ and $Q(0,f,f,0)$. We find that P is on the surface $m_2 = 1-n_2^2$, and with $m_2 = n_2 = f$, we find: $f = \frac{-1 + \sqrt{5}}{2} = 0.618$ and Q is on the surface $m_1-m_2 = 1 - (n_1-n_2)^2$ and with: $n_1=0$, $n_2=f$, $m_1=f$, $m_2=0$, we find $f = \frac{-1 + \sqrt{5}}{2} = 0.618$.

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